

Exact mathematical models of a unified quantum theory; Static and expanding micro universes

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Abstract

In this paper such Riemann metrics are established whose Laplace-Beltrami operators are identical to familiar Hamilton operators of elementary particle systems. Such metrics are the natural positive definite invariant metrics defined on two-step nilpotent Lie groups. The corresponding wave and Schrödinger operators emerge in the Laplacians both of the static and solvable extensions of these nilpotent groups. The latter manifolds are endowed with natural invariant indefinite metric of Lorentz signature. Thus, these new exact mathematical models provide a relativistic theory for elementary particles.

This theory establishes infinitely many non-equivalent models for which even classification is possible. The particle systems attached to them behave exactly like their relatives introduced by the familiar standard model of elementary particle physics. Although there are strong connections between the two theories, the new mathematical models were independently discovered. They appeared, originally, in those works of this author where isospectral Riemann manifolds with different local geometries have been constructed. Their strong connection to quantum theory, which seems to be completely unknown in the literature, was realized by some years later.

One of the most important new features of this new theory is that the electromagnetic, the weak-nuclear, and the strong-nuclear forces emerge in a unified way. The main unifying idea is that these forces can be described by the eigenfunctions of the very same Laplacian such that the distinct forces emerge on distinct invariant subspaces of this common quantum operator. There are also other bonds realized which make the connection among these three fundamental forces much more strong. The missing fourth fundamental force, gravitation, is not discussed in this paper.

On the solvable extensions the new models look like Friedmann's expanding universe being adopted to the microscopic level. Like the macroscopic one, also these microscopic expanding models obey Hubble's law. The microscopic models, however, offer much more complex structures with much more subtle explanations for some of those phenomena which have originally been clarified by the Friedmann model. For instance, it is rigorously established in this paper that, although the Riemann spaces in the mathematical models are not spatially isotropic in general, yet they are always spectrally isotropic. It follows that the radiation, which is experimentally known in cosmology, must also be isotropic. This statement actually contrasts the widely accepted view that the isotropic cosmic radiation is the chief verification of the spatial-isotropy assumed by Friedmann in building up his theory.

1 Introduction

By assuming the metric in the form $ds^2 = a(t)^2 ds_3^2 - dt^2$, Friedmann (1922) built up his relativistic universe on two assumptions: the universe looks identical in whichever direction we look, and this would also be true if we were observing it anywhere else. By the two differential equations imposed by the Einstein equation upon the function $a(t)$, he showed then that, instead of being static, the universe is expanding. Without knowing Friedmann's theoretical prediction, this phenomenon was actually discovered by Edwin Hubble (1929) several years later.

This phenomenon is pointed out in this paper in quite a different situation. Independently from Friedmann's theory, this one has been evolved from those works of the author where isospectral manifolds with different local geometries were constructed on two different types of manifolds, called Z-torus bundles (alias Z-crystals) and Z-ball bundles, respectively [Sz1]-[Sz4]. These bundles are constructed by means of nilpotent Lie groups as well as their solvable extensions such that one considers tori resp. balls in the center (called also Z-space) of the nilpotent group. Both the nilpotent and extended groups are endowed with appropriate natural invariant metrics.

Surprisingly enough, the Laplacians on the nilpotent groups (endowed always with invariant positive definite Riemann metrics) are nothing but the familiar Hamilton operators corresponding to particle-antiparticle systems. On the two types of manifolds, the represented particles can be distinguished as follows. On Z-crystals, the Laplacian represents particles having no interior, where it actually appears as a Ginsburg-Landau-Zeeman operator of a system of electrons, positrons, and electron-positron-neutrinos. The particles represented by Z-ball bundles do have interior and the Laplacian decomposes into an exterior Ginsburg-Landau-Zeeman operator and an interior spin operator by which the weak-force and the strong-force interactions can be described, respectively. These nuclear forces are very different from the electromagnetic force emerging in the Laplacian of Z-crystals. The weak nuclear force explains the beta decay, while the strong force keeps the parts of atomic nuclei together. Yet, the Z-crystal-Laplacian and the weak-force-Laplacian can be led back to the very same radial Ginsburg-Landau-Zeeman operator. This phenomenon is consistent with the Weinberg-Salam theory of beta decays, which unified the weak force with the electromagnetic force.

There are two ways to introduce relativistic time on these models. The static model is constructed by the Cartesian product of the nilpotent group with \mathbb{R} . The latter component becomes the time axis regarding the natural Lorenz-indefinite metric. According to the type of model being extended, the

Laplacian is the sum of Schrödinger and electron-positron-neutrino, resp., weak-nuclear and strong-nuclear wave operators. The last two wave operators appear for particles having interior. They are further decomposed into W- and Z-operators which are analogous to the electron-positron and electron-positron-neutrino wave operators.

Relativistic time can be introduced also by solvable extension, which also increases the dimension of the nilpotent group by 1. The new axis, which is just a half-line \mathbb{R}_+ , can also be used as time-axis for introducing a natural relativistic metric on these extensions. Contrary to the static case, in this way one defines expanding models obeying Hubble's law, furthermore, the Laplacian decomposes into expanding Schrödinger and electron-positron-neutrino, resp., weak and strong wave operators and the corresponding W- and Z-operators.

It is a well known experimental fact that, even though the universe is expanding, there is no expansion measured on small scale level. Thus the question arises if the expanding solvable models describe real existing microscopic world. Fortunately this question can positively be answered. In fact, despite of the expansion, the particles must not be expanding. The reason explaining this paradoxical phenomenon is that, defined by the angular momentum and spin operators, also these mathematical models correspond constant magnetic fields to the particles, and, due to the expansion, the change of these fields induces electromagnetic fields, which are completely radiated out from the system, keeping both the magnetic fields and the spectra of the particle-systems constant. There is also mathematically established in this paper that this radiation must be isotropic, meaning that it is the same whichever direction is measured from. Thus the size and several other constants of particles must not be changing even according to the expanding solvable model. Actually this model gives a new explanation for the presence of an isotropic radiation in the space.

Expanding on large scale but being stationary on small scale is a well known phenomenon, which, without the above explanation, could have been a major argument against the physical reality of the solvable extensions. According to physical experiments, although far distant clusters of galaxies move very rapidly away from us, the solar system is not expanding, nor is our galaxy or the cluster of galaxies to which it belongs. This stagnancy is even more apparent on microscopic level, where, for instance, the spectroscopic investigations of the light arriving from far distance galaxies confirm that the spectrum of hydrogen atom is the same today than it was macro billions of years ago.

The existence of isotropic background radiation is also well known which was measured, first, by Arno Penzias and Robert Wilson, in 1965. It is believed, today, that these radiations are travelling to us across most of the observable universe, thus the radiation isotropy proves that the universe must be the same in every direction, if only on a large scale. This phenomenon is considered as a remarkably accurate confirmation of Friedmann's first assumption. Our expanding model provides a much more subtle conclusion, however: This background radiation must be isotropic even if it arrives to us from very near distances. Moreover, it holds true also on non-isotropic spaces.

This exact mathematical model is not derived from the standard model of elementary particles, which theory is based on a non-Abelian gauge theory where the basic objects are Yang-Mills connections defined on principal fibre bundles having structure group $SU(3)$. Contrary to these gauge theories, in our case all physical quantities are defined by invariant Riemann metrics living on nilpotent resp. solvable groups. It will also be pointed out that no regular gauge-group exist on these models regarding of which these objects are gauge invariant. Yet, there is a bridge built up between the two theories, which explains why the particles introduced by the two distinct models exhibit the very same physical features. This bridge can be regarded as a correspondence principle associating certain Riemann metrics to the Yang-Mills models of elementary particles.

The key point about the new exact mathematical model is that the center of the nilpotent group makes room to describe also the rich "inner life" of particles, which is known both experimentally and by theories explaining these experimental facts. This "inner life" is displayed by the de Broglie waves which appear in a new form in this new situation such that they are written up in terms of the Fourier transforms performed only on the center of the nilpotent group. By this reason, they are called Z-Fourier transforms, which are defined on the two types of models accordingly. On Z-crystals, where there is no "inner life", it is nothing but the discrete Z-Fourier transform defined by the Z-lattice by which the Z-crystals are introduced. On Z-ball bundles, however, in order to obey the boundary conditions, more complicated so called twisted Z-Fourier transforms are introduced.

The action of the very same Laplacian appears quite differently on these different function spaces. On Z-crystals, where there are no Z-boundary conditions involved, the strong nuclear forces do not appear either. In this case, the eigenfunctions arise as eigenfunctions of Ginsburg-Landau-Zeeman operators. By this reason, they are called electromagnetic eigenfunctions.

On Z-crystals, the theory corresponds to quantum electrodynamics (QED), while on Z-ball bundles it relates to quantum chromodynamics (QCD).

In fact, on Z-ball bundles, due to the the Z-boundary conditions, the Laplacian appears in a much more complex form exhibiting both the weak and strong forces. More precisely, the weak force eigenfunctions satisfying a given boundary condition are defined by the eigenfunctions of the exterior Ginsburg-Landau-Zeeman operator introduced above for particles having interior. Although there are numerous differences between this exterior Ginsburg-Landau-Zeeman operator and the original GLZ-operator defined on Z-crystals, they are both reduced to the very same radial operator acting on radial functions. As a result, from the point of view of the elements of the spectrum, they are the same operators. This is the mathematical certification of the Weinberg-Salam theory which unified the weak interaction with the electromagnetic force. The strong force eigenfunctions are defined by the eigenfunctions of the inner spin operator. All these forces reveal the very same strange properties which are described in QCD. This is how a unified theory for the three: 1.) electromagnetic, 2.) weak, and 3.) strong nuclear forces is established in this paper. The only elementary force missing from this list is the gravitation, which is not discussed in this paper.

This very complex physical-mathematical theory can clearly be evolved just gradually. In order to understand the physical contents of the basic objects appearing in new forms in this new approach, first, those parts of the classical quantum theory are reviewed which are necessary to grasp these renewed versions of these basic concepts. Then, after introducing the basic mathematical objects on 2-step nilpotent Lie groups, several versions of the Z-Fourier transform will be studied. They are the basic tools both for introducing the de Broglie waves in a new explicit form and developing the theory unifying the three fundamental forces. Besides explicit eigenfunction computations, there is pointed out in this part that the Laplacian on Z-crystals is nothing but the Ginsburg-Landau-Zeeman operator of a system of electrons, positrons, and electron-positron-neutrinos. Furthermore, on the Z-ball models, it is the sum of the exterior Ginsburg-Landau-Zeeman operator and the interior spin operator by which the strong force interaction can be established.

Then, relativistic time is introduced and both static and expanding models are established. The Laplacian on these space-time manifolds appears as the sum of wave operators belonging to the particles the system consists of. The paper is concluded by pointing out the spectral isotropy in the most general situations. Since the Riemann metrics attached to the particle systems are not isotropic in general, this statement points out a major

difference between our model and Friedmann's cosmological model where the isotropy of the space is one of his two assumptions. Our statement says that radiation isotropy holds true also on non-isotropic spaces and the two isotropy-concepts are by no means equivalent.

2 Basics of classical quantum theory.

In this sections three topics of classical quantum theory are reviewed. The first one describes the elements of de Broglie's theory associating waves to particles. The second resp. third ones are surveys on meson theory resp. Ginsburg-Landau-Zeeman and Schrödinger operators of charged particles.

2.1 Wave-particle association.

In quantum theory, a particle with energy E and momentum \mathbf{p} is associated with a wave, $Ae^{i(K \cdot Z - \omega t)}$, where $K = (2\pi/\lambda)\mathbf{n}$ is the wave vector and \mathbf{n} is the wave normal. These quantities yield the following relativistically invariant relations.

For light quanta the most familiar relations are

$$(1) \quad E = \hbar\omega \quad , \quad \mathbf{p} = \hbar K,$$

where the length of the wave vector yields also the following equations:

$$(2) \quad \mathbf{k} = |K| = \frac{\omega}{c}, \quad \mathbf{k}^2 = \frac{\omega^2}{c^2}, \quad \text{and} \quad |\mathbf{p}| = p = \frac{E}{c}, \quad \mathbf{p}^2 = \frac{E^2}{c^2}.$$

For a material particle of rest mass m , the fundamental relation is

$$(3) \quad \frac{E}{c} = \sqrt{\mathbf{p}^2 + m^2 c^2},$$

which can be established by the well known equations

$$(4) \quad E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}$$

of relativistic particle mechanics.

The idea of de Broglie was that (1) should also be valid for a material particle such that (2) must be replaced by

$$(5) \quad \sqrt{\mathbf{k}^2 + \frac{m^2 c^2}{\hbar^2}} = \frac{\omega}{c}, \quad \mathbf{k}^2 + \frac{m^2 c^2}{\hbar^2} = \frac{\omega^2}{c^2}.$$

In this general setting, $m = 0$ corresponds to the light.

In wave mechanics, de Broglie's most general wave packets are represented by the Fourier integral formula:

$$(6) \quad \psi(Z, t) = \int \int \int A(K_1, K_2, K_3) e^{i(\langle K, Z \rangle - \omega t)} dK_1 dK_2 dK_3,$$

where ω is given by (5). In other words, a general wave appears as superposition of the above plane waves. Instead of the familiar X , the vectors from the 3-space, \mathbb{R}^3 , are denoted here by Z , indicating that the reformulated de Broglie waves will be introduced in the new theory in terms of the so called twisted Z-Fourier transform, which is performed just on the center (alias Z-space) of the nilpotent group. The X-space of a nilpotent group is complement to the Z-space and the integration in the formula of twisted Z-Fourier transform does not apply to the X-variable. It applies just to the Z-variable. The above denotation is intended to help to understand the renewed de Broglie waves more easily.

The above wave function, ψ , satisfies the relativistic scalar wave equation:

$$(7) \quad (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \psi(Z, t) = \frac{m^2 c^2}{\hbar^2} \psi(Z, t),$$

which statement can be seen by substituting (6) into this equation. Then (5) implies (7), indeed. According to this equation, the wave function is an eigenfunction of the wave operator with eigenvalue $m^2 c^2 / \hbar^2$. By this observation we get that the spectrum of the wave operator is continuous and the multiplicity of each eigenvalue is infinity.

The Fourier integral formula (6) converts differential operators to multiplication operators. Namely we have:

$$(8) \quad \frac{\partial}{\partial Z_j} \sim iK_j \quad , \quad \frac{\partial}{\partial t} \sim i\omega.$$

These correspondences together with (1) yield the translational key:

$$(9) \quad -i\hbar \frac{\partial}{\partial Z_j} \sim p_j \quad , \quad i\hbar \frac{\partial}{\partial t} \sim E$$

between the classical quantities \mathbf{p} and E of classical mechanics and the operators of wave mechanics.

In his lectures on physics [P] (Vol. 5, Wave mechanics, pages 3-4), Pauli describes the transition from the above relativistic theory to the non-relativistic approximation as follows. In mechanics, for $v \ll c$ and $p \ll mc$, we have

$$(10) \quad \frac{E}{c} = \sqrt{\mathbf{p}^2 + m^2 c^2} \sim mc(1 + \frac{1}{2} \frac{p^2}{m^2 c^2} + \dots) = \frac{1}{c}(mc^2 + \frac{1}{2} \frac{p^2}{m} + \dots).$$

From (5) we also obtain

$$(11) \quad \omega = \frac{E}{\hbar} = \frac{mc^2}{\hbar} + \frac{\hbar}{2m}k^2 + \dots,$$

where $E = mc^2 + E_{kin}$ and $E_{kin} = p^2/2m$. The non-relativistic wave

$$(12) \quad \tilde{\psi}(Z, t) = \int \int \int A(K_1, K_2, K_3) e^{i(\langle K, Z \rangle - \tilde{\omega}t)} dK_1 dK_2 dK_3,$$

is defined in terms of

$$(13) \quad \tilde{\omega} = \frac{\hbar}{2m}\mathbf{k}^2 = \omega - \frac{mc^2}{\hbar},$$

which relates to the relativistic wave function by the formula:

$$(14) \quad \psi(Z, t) = e^{-\frac{imc^2}{\hbar}t} \tilde{\psi}(Z, t).$$

Substitution into (7) yields then:

$$(15) \quad \nabla^2 \tilde{\psi} + \frac{m^2 c^2}{\hbar^2} \tilde{\psi} + 2 \frac{i m}{\hbar} \frac{\partial \tilde{\psi}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \tilde{\psi}}{\partial t^2} = \frac{m^2 c^2}{\hbar^2} \tilde{\psi},$$

which is nothing but the non-relativistic wave equation:

$$(16) \quad \nabla^2 \tilde{\psi} + i \frac{2m}{\hbar} \frac{\partial \tilde{\psi}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \tilde{\psi}}{\partial t^2} = 0.$$

The imaginary coefficient ensures that there is no special direction in time. This equation is invariant under the transformations $t \rightarrow -t$ and $\tilde{\psi} \rightarrow \tilde{\psi}^*$, whereby $\tilde{\psi}\tilde{\psi}^*$, where $*$ means conjugation of complex numbers, remains unchanged (this conjugation will be denoted by $\bar{\psi}$ later on). According to quantum theory, the physically measurable quantity is not the wave functions ψ or $\tilde{\psi}$, but the probability densities $\psi\psi^*$ resp. $\tilde{\psi}\tilde{\psi}^*$.

2.2 Meson theory.

The above solutions of the wave equation strongly relate to the theory of nuclear forces and mesons. To review this field, we literally quote Hideki Yukawa's Nobel Lecture "Meson theory in its developments", delivered on December 12, 1949. Despite the fact that elementary particle physics went through enormous developments since 1949, this review has been chosen because it is highly suggestive regarding the physical interpretations of the mathematical models developed in this paper:

"The meson theory started from the extension of the concept of the field of force so as to include the nuclear forces in addition to the gravitational and electromagnetic forces. The necessity of introduction of specific nuclear

forces, which could not be reduced to electromagnetic interactions between charged particles, was realized soon after the discovery of the neutron, which was to be bound strongly to the protons and other neutrons in the atomic nucleus. As pointed out by Wigner¹, specific nuclear forces between two nucleons, each of which can be either in the neutron state or the proton state, must have a very short range of the order of 10-13 cm, in order to account for the rapid increase of the binding energy from the deuteron to the alphaparticle. The binding energies of nuclei heavier than the alpha-particle do not increase as rapidly as if they were proportional to the square of the mass number A, i.e. the number of nucleons in each nucleus, but they are in fact approximately proportional to A. This indicates that nuclear forces are saturated for some reason. Heisenberg² suggested that this could be accounted for, if we assumed a force between a neutron and a proton, for instance, due to the exchange of the electron or, more generally, due to the exchange of the electric charge, as in the case of the chemical bond between a hydrogen atom and a proton. Soon afterwards, Fermi³ developed a theory of beta-decay based on the hypothesis by Pauli, according to which a neutron, for instance, could decay into a proton, an electron, and a neutrino, which was supposed to be a very penetrating neutral particle with a very small mass.

This gave rise, in turn, to the expectation that nuclear forces could be reduced to the exchange of a pair of an electron and a neutrino between two nucleons, just as electromagnetic forces were regarded as due to the exchange of photons between charged particles. It turned out, however, that the nuclear forces thus obtained was much too small⁴, because the betadecay was a very slow process compared with the supposed rapid exchange of the electric charge responsible for the actual nuclear forces. The idea of the meson field was introduced in 1935 in order to make up this gaps. Original assumptions of the meson theory were as follows:

I. The nuclear forces are described by a scalar field U, which satisfies the wave equation

$$(17) \quad \left(\frac{\partial^2}{\partial Z_1^2} + \frac{\partial^2}{\partial Z_2^2} + \frac{\partial^2}{\partial Z_3^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \kappa^2 \right) U = 0$$

in vacuum, where κ is a constant with the dimension of reciprocal length. Thus, the static potential between two nucleons at a distance r is proportional to $\exp(-\kappa r)/r$, the range of forces being given by $1/\kappa$.

II. According to the general principle of quantum theory, the field U is inevitably accompanied by new particles or quanta, which have the mass

$$(18) \quad \mu = \frac{\kappa \hbar}{c}$$

and the spin 0, obeying Bose-Einstein statistics. The mass of these particles can be inferred from the range of nuclear forces. If we assume, for instance, $x = 5 \times 10^{12} \text{cm}^{-1}$, we obtain $\mu \sim 200m_e$, where m_e is the mass of the electron.

III. In order to obtain exchange forces, we must assume that these mesons have the electric charge $+e$ or $-e$, and that a positive (negative) meson is emitted (absorbed) when the nucleon jumps from the proton state to the neutron state, whereas a negative (positive) meson is emitted (absorbed) when the nucleon jumps from the neutron to the proton. Thus a neutron and a proton can interact with each other by exchanging mesons just as two charged particles interact by exchanging photons. In fact, we obtain an exchange force of Heisenberg type between the neutron and the proton of the correct magnitude, if we assume that the coupling constant g between the nucleon and the meson field, which has the same dimension as the elementary charge e , is a few times larger than e .

However, the above simple theory was incomplete in various respects. For one thing, the exchange force thus obtained was repulsive for triplet S-state of the deuteron in contradiction to the experiment, and moreover we could not deduce the exchange force of Majorana type, which was necessary in order to account for the saturation of nuclear forces just at the alpha-particle. In order to remove these defects, more general types of meson fields including vector, pseudoscalar and pseudovector fields in addition to the scalar fields, were considered by various authors⁶. In particular, the vector field was investigated in detail, because it could give a combination of exchange forces of Heisenberg and Majorana types with correct signs and could further account for the anomalous magnetic moments of the neutron and the proton qualitatively. Furthermore, the vector theory predicted the existence of noncentral forces between a neutron and a proton, so that the deuteron might have the electric quadrupole moment. However, the actual electric quadrupole moment turned out to be positive in sign, whereas the vector theory anticipated the sign to be negative. The only meson field, which gives the correct signs both for nuclear forces and for the electric quadrupole moment of the deuteron, was the pseudoscalar field⁷. There was, however, another feature of nuclear forces, which was to be accounted for as a consequence of the meson theory. Namely, the results of experiments on the scattering of protons by protons indicated that the type and magnitude of interaction between two protons were, at least approximately, the same as those between a neutron and a proton, apart from the Coulomb force. Now the interaction between two protons or two neutrons was obtained only if we took into account the terms proportional to g^4 , whereas that between a neutron and a proton was proportional to g^2 , as long as we were considering charged mesons alone. Thus it seemed necessary to assume further:

IV. In addition to charged mesons, there are neutral mesons with the mass either exactly or approximately equal to that of charged mesons. They must also have the integer spin, obey Bose-Einstein statistics and interact with nucleons as strongly as charged mesons. This assumption obviously increased the number of arbitrary constants in meson theory, which could be so adjusted as to agree with a variety of experimental facts. These experimental facts could-not be restricted to those of nuclear physics in the narrow sense, but was to include those related to cosmic rays, because we expected that mesons could be created and annihilated due to the interaction of cosmic ray particles with energies much larger than cosmic rays in 1937⁸ was a great encouragement to further developments of meson theory. At that time, we came naturally to the conclusion that the mesons which constituted the main part of the hard component of cosmic rays at sea level was to be identified with the mesons which were responsible for nuclear force⁹. Indeed, cosmic ray mesons had the mass around $200m_e$ as predicted and moreover, there was the definite evidence for the spontaneous decay, which was the consequence of the following assumption of the original meson theory :

V. Mesons interact also with light particles, i.e. electrons and neutrinos, just as they interact with nucleons, the only difference being the smallness of the coupling constant g in this case compared with g . Thus a positive (negative) meson can change spontaneously into a positive (negative) electron and a neutrino, as pointed out first by Bhabha¹⁰. The proper lifetime, i.e. the mean lifetime at rest, of the charged scalar meson, for example, is given by

$$(19) \quad \tau_0 = 2 \left(\frac{\hbar c}{(g')^2} \right) \left(\frac{\hbar}{\mu c^2} \right)$$

For the meson moving with velocity ν , the lifetime increases by a factor $1/\sqrt{1 - (\nu/c)^2}$ due to the well-known relativistic delay of the moving clock. Although the spontaneous decay and the velocity dependence of the lifetime of cosmic ray mesons were remarkably confirmed by various experiments¹¹, there was an undeniable discrepancy between theoretical and experimental values for the lifetime. The original intention of meson theory was to account for the beta-decay by combining the assumptions III and V together. However, the coupling constant g , which was so adjusted as to give the correct result for the beta-decay, turned out to be too large in that it gave the lifetime τ_0 of mesons of the order of 10^{-8} sec , which was much smaller than the observed lifetime $2 \times 10^{-6} \text{ sec}$. Moreover, there were indications, which were by no means in favour of the expectation that cosmic-ray mesons interacted strongly with nucleons. For example, the observed crosssection of scattering

of cosmic-ray mesons by nuclei was much smaller than that obtained theoretically. Thus, already in 1941, the identification of the cosmic-ray meson with the meson, which was supposed to be responsible for nuclear forces, became doubtful. In fact, Tanikawa and Sakata¹² proposed in 1942 a new hypothesis as follows: The mesons which constitute the hard component of cosmic rays at sea level are not directly connected with nuclear forces, but are produced by the decay of heavier mesons which interacted strongly with nucleons. However, we had to wait for a few years before this two-meson hypothesis was confirmed, until 1947, when two very important facts were discovered. First, it was discovered by Italian physicists that the negative mesons in cosmic rays, which were captured by lighter atoms, did not disappear instantly, but very often decayed into electrons in a mean time interval of the order of 10^{-6} sec . This could be understood only if we supposed that ordinary mesons in cosmic rays interacted very weakly with nucleons. Soon afterwards, Powell and others¹⁴ discovered two types of mesons in cosmic rays, the heavier mesons decaying in a very short time into lighter mesons. Just before the latter discovery, the two-meson hypothesis was proposed by Marshak and Bethe¹⁵ independent of the Japanese physicists above mentioned. In 1948, mesons were created artificially in Berkeley¹⁶ and subsequent experiments confirmed the general picture of two-meson theory. The fundamental assumptions are now¹⁷

(i) The heavier mesons, i.e. n -mesons with the mass m_π , about $280m_e$ interact strongly with nucleons and can decay into lighter mesons, i.e. π -mesons and neutrinos with a lifetime of the order of 10^{-8} sec ; π -mesons have integer spin (very probably spin 0) and obey Bose-Einstein statistics. They are responsible for, at least, a part of nuclear forces. In fact, the shape of nuclear potential at a distance of the order of $\hbar/m_\pi c$ or larger could be accounted for as due to the exchange of π -mesons between nucleons.

(ii) The lighter mesons, i.e. μ -mesons with the mass about $210m_e$ are the main constituent of the hard component of cosmic rays at sea level and can decay into electrons and neutrinos with the lifetime $2 \times 10^{-6} \text{ sec}$. They have very probably spin $\frac{1}{2}$ and obey Fermi-Dirac statistics. As they interact only weakly with nucleons, they have nothing to do with nuclear forces. Now, if we accept the view that π -mesons are the mesons that have been anticipated from the beginning, then we may expect the existence of neutral π -mesons in addition to charged p -mesons. Such neutral mesons, which have integer spin and interact as strongly as charged mesons with nucleons, must be very unstable, because each of them can decay into two or three photons¹⁸. In particular, a neutral meson with spin 0 can decay into two photons and the lifetime is of the order of 10^{-14} sec or even less than that. Very recently,

it became clear that some of the experimental results obtained in Berkeley could be accounted for consistently by considering that, in addition to charged π -mesons, neutral ν -mesons with the mass approximately equal to that of charged π -mesons were created by collisions of high-energy protons with atomic nuclei and that each of these neutral mesons decayed into two mesons with the lifetime of the order of 10^{-13} sec or less¹⁹. Thus, the neutral mesons must have spin 0. In this way, meson theory has changed a great deal during these fifteen years. Nevertheless, there remain still many questions unanswered. Among other things, we know very little about mesons heavier than π -mesons. We do not know yet whether some of the heavier mesons are responsible for nuclear forces at very short distances. The present form of meson theory is not free from the divergence difficulties, although recent development of relativistic field theory has succeeded in removing some of them. We do not yet know whether the remaining divergence difficulties are due to our ignorance of the structure of elementary particles themselves²⁰. We shall probably have to go through another change of the theory, before we shall be able to arrive at the complete understanding of the nuclear structure and of various phenomena, which will occur in high energy regions.

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2.3 Zeeman and Schrödinger operators of electrons.

In physics, the classical Zeeman operator of a charged particle is:

$$(20) \quad -\frac{\hbar^2}{2m}\Delta_{(x,y)} - \frac{\hbar e B}{2m c i} D_z \bullet + \frac{e^2 B^2}{8m c^2} (x^2 + y^2) + eV.$$

Originally, this operator is considered on the 3-space expressed in terms of 3D Euclidean Laplacian and 3D magnetic dipole momentum operators. The latter operators are the first ones where a preliminary version of spin concept

appears in the history of physics. This is the so called exterior or orbiting spin associated with the 3D angular momentum

$$(21) \quad \mathbf{P} = (P_1, P_2, P_3) = \frac{1}{\hbar} \mathbf{Z} \times \mathbf{p} \quad \text{where}$$

$$(22) \quad P_1 = \frac{1}{\hbar} (Z_2 p_3 - Z_3 p_2) = \frac{1}{i} \left(Z_2 \frac{\partial}{\partial Z_3} - Z_3 \frac{\partial}{\partial Z_2} \right) \dots$$

Note that the above 2D operator keeps only component P_3 of this angular momentum. The components of the complete 3D angular momentum obeys the commutation relations:

$$(23) \quad [P_1, P_2] = iP_3, \quad [P_1, P_3] = -iP_2, \quad [P_2, P_3] = iP_1.$$

In the mathematical models, the particles are orbiting in complex planes determined by the complex structures associated with the component angular momenta P_j , thus the 2D version plays more important role in this paper than the 3D version. The 2D operator is obtained from the 3D operator by omitting P_1 and P_2 and by restricting the rest part onto the (x, y) plane. If Coulomb potential V is omitted, it is nothing but the *Ginsburg-Landau-Zeeman operator* of a charged particle orbiting on the (x, y) -plane in a constant magnetic field directed toward the z -axis. The *magnetic dipole momentum operator*, which is the term involving $D_z \bullet := x\partial_y - y\partial_x$ and which is associated with the *angular momentum operator* $\hbar D_z$, commutes with the rest part, \mathbf{O} , of the operator, therefore, splitting the spectral lines of \mathbf{O} . The *Zeeman effect* is explained by this fine structure of the Zeeman operator.

The Hamilton operator represents the total energy of a given physical system. More precisely, the eigenvalues of this operator are the discrete (quantized) energy values which can be assumed by the system. Thus, correspondence (9) implies Schrödinger's wave equation

$$(24) \quad -\left(\frac{\hbar^2}{2m} \Delta_{(x,y)} - \frac{\hbar e B}{2mc i} D_z \bullet + \frac{e^2 B^2}{8mc^2} (x^2 + y^2) - eV \right) \psi = i\hbar \frac{\partial \psi}{\partial t}$$

of an electron orbiting in the (x, y) -plane.

As it well known, Schrödinger discovered first the relativistic equation which is a second order differential operator regarding the t -variable. By the time that Schrödinger came to publish this equation, it had already been independently rediscovered by O. Klein and W. Gordon. This is why it is usually called Klein-Gordon equation. Numerous problems had arise regarding this equation. Schrödinger became discouraged because it gave the wrong fine structure for hydrogen. Some month later he realized, however, that the non-relativistic approximation to his relativistic equation was of

value even if the relativistic equation was incorrect. This non-relativistic approximation is the familiar Schrödinger equation.

Dirac also had great concerns about the Klein-Gordon equation. His main objection was that the probabilistic quantum theory based on this equation produced negative probabilities. Actually, the elimination of this problem led Dirac to the discovery of his relativistic electron equation. By this theory, however, proper probabilistic theory can be developed only on the relativistic space-time. This feature was strongly criticized by Pauli, according to whom such theory makes sense only on the space.

The journey toward an understanding of the nature of spin and its relationship to statistics has been taking place on one of the most difficult and exciting routes [B, Tom]. Although the Schrödinger wave equation gives excellent agreement with experiment in predicting the frequencies of spectral lines, small discrepancies are found, which can be explained only by adding an intrinsic angular momentum to its usual orbital angular momentum of the electron that acts as if it came from a spinning solid body. The pioneers of developing this concept were Sommerfeld, Landé, and Pauli. They found that agreement with the Stern-Gerlach experiment proving the existence of Zeeman effect can be obtained by assuming that the magnitude of this additional angular momentum was $\hbar/2$. The magnetic moment needed to obtain agreement was, however, $e\hbar/2mc$, which is exactly the same as that arising from an orbital angular momentum of \hbar . The gyromagnetic ratio, that is, the ratio of magnetic moment to angular momentum is therefore twice as great for electron spin as it is for orbital motion.

Many efforts were made to connect this intrinsic angular momentum to an actual spin of the electron, considered it as a rigid body. In fact, the gyromagnetic ratio needed is exactly that which would be obtained if the electron consisted of a uniform spherical shell spinning about a definite axis. The systematic development of such a theory met, however, with such great difficulties that no one was able to carry it through to a definite conclusion. Somewhat later, Dirac derived his above mentioned relativistic wave equation for the electron, in which the spin and charge were shown to be bound up in a way that can be understood only in connection with the requirements of relativistic invariance. In the non-relativistic limit, however, the electron still acts as if it had an intrinsic angular momentum of $\hbar/2$. Prior to the Dirac equation, this non-relativistic theory of spin was originally developed by Pauli.

Finally, we explain yet why the Coulomb operator does not appear in this paper and how can it be involved into the further investigations. The present

ignorance is mainly due to the fact that also the Hamilton operator (Laplacian) on nilpotent groups involves no Coulomb potential. There appear, instead, nuclear potentials like those Yukawa described in meson theory. An other major distinguishing feature is that this Laplacian (Hamilton operator) includes also terms corresponding to the electron-positron-neutrino, which, by the standard model, is always something of a silent partner in an electron-positron-system, because, being electrically neutral, it ignores not only the nuclear force but also the electromagnetic force. Although the operator corresponding to this silent partner will be established by computations evolved by Pauli to determine the non-relativistic approximation, all these operators appear in the new theory as relativistic operators complying with Einstein's equation of general relativity.

The Coulomb force can be considered just later, after developing certain explicit spectral computations. This spectral theory includes also a spectral decomposition of the corresponding L^2 -function spaces such that the subspaces appearing in this decomposition are invariant with respect to the actions both of the Hamilton operator and the complex Heisenberg group representation. Also the latter representation is naturally inbuilt into these mathematical models.

The complications about the Coulomb operator are due to the fact that these invariant subspaces (called also zones) are not invariant regarding the Coulomb's multiplicative operator [Sz5]- [Sz7]. In order to extend the theory also to electric fields, the Coulomb operator must be modified such that also this operator leaves the zones invariant. Such natural zonal Coulomb operator can be defined for a particular zone such that, for a given function ψ from the zone, function $V\psi$ is projected back to the zone. This modified Coulomb operator is the correct one which must be added to the Laplace operator on the nilpotent Lie group in order to have a relevant unified electro-magnetic particle theory.

However, this modified Coulomb force is externally added and not naturally inbuilt into the Laplacian (Hamiltonian) of the nilpotent Lie group. In order to construct such Riemann manifolds whose Laplacian unifies the Ginsburg-Landau-Zeeman +neutrino+ nuclear operators also with an appropriate Coulomb operator, the mathematical models must be further developed such that, instead of nilpotent Lie groups, one considers general nilpotent-type Riemann manifolds and their solvable-type extensions. This generalization of the theory, which is similar to passing from special relativity to the general one, will be the third step in developing this theory.

3 Launching the mathematical particle theory.

The physical features of elementary particles are most conspicuously exhibited also by certain Riemann manifolds. A demonstration of this apparent physical content present in these abstract mathematical structures is, for instance, that the classical Hamilton and Schrödinger operators of elementary particle systems appear as Laplace-Beltrami operators defined on these manifolds. Thus, these abstract structures are really deeply inbuilt into the very fabric of the physical world which can serve also as fundamental tools for building up a comprehensive unified quantum theory.

Yet, this physical content of these particular mathematical structures has never been recognized in the literature so far. The ignorance is probably due to the fact that this new theory is not in a direct genetic relationship with fundamental theories such as the Gell-Mann~Ne'eman theory of quarks by which the standard model of elementary particles has been established. Neither is a direct genetic connection to the intensively studied super string theory, which, by many experts, is thought to be the first viable candidate ever for a unified quantum field theory of all of the elementary particles and their interactions which had been provisionally described by the standard model. Actually, the relationship between the two approaches to elementary particle physics is more precisely described by saying that there are both strong connections and substantial differences between the standard and our new models. In order to clearly explain the new features of the new model, we start with a brief review of the standard model. A review of string theory is, however, beyond the scope of this article.

3.1 A rudimentary review of the standard model.

The Gell-Mann~Ne'eman theory [GN] of quantum chromodynamics (QCD) grew out, in the 60's, from Yang-Mills' [YM] non-Abelian gauge theory where the gauge group was taken to be the $SU(2)$ group of isotopic spin rotations, and the vector fields analogous to the photon field were interpreted as the fields of strongly-interacting vector mesons of isotopic spin unity. QCD is also a non-Abelian gauge theory where, instead of $SU(2)$, the symmetry group is $SU(3)$. This model adequately described the rapidly growing number of elementary particles by grouping the known baryons and mesons in various irreducible representations of this gauge group.

The most important difference between this and the new mathematical model is that the Yang-Mills theories do not introduce a (definite or indefinite) Riemann metric with the help of which all other physical objects are

defined. They rather describe the interactions by Lagrange functions which contain more than a dozen arbitrary constants, including those yielding the various masses of the different kinds of particles. What makes this situation even worse is that all these important numbers are incalculable in principle. Under such circumstances the natural Hamilton and wave operators of particles can not emerge as Laplacians on Riemann manifolds. This statement is the main attraction on the new models.

To speak more mathematically, the fundamental structures for non-Abelian gauge theories are principal fibre bundles over Minkowski space with compact non-Abelian structure group $SU(n)$ on which a potential is defined by a connection A , with components A_μ , in the Lie algebra $su(n)$. The field is the curvature whose components are $F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The most straightforward generalization of Maxwell's equations are the Yang-Mills equations $dF = 0$ and $d^*F = 0$, where d and d^* are covariant derivatives. Gauge theories possess an infinite-dimensional symmetry group given by functions $g : M \rightarrow SU(n)$ and all physical or geometric properties are gauge invariant.

To specify a physical theory the usual procedure is to define a Lagrangian. In quantum chromodynamics (QCD) such Lagrangian is to be chosen which is capable to portray the elementary particles in the following very rich complexity: The neutron and proton are composite made of quarks. There are quark of six type, or "flavors", the u , c , and t quarks having charge $2/3$, and the d , s , and b quarks having charge $-1/3$ (these denotations are the first letters of words: up, down, charm, strange, top, and bottom). Quarks of each flavor come in three "colors" which furnish the defining representation **3** of the $SU(3)$ gauge group.

Quarks have the remarkable property of being permanently trapped inside "white" particles such as neutron, proton, baryon and meson. Baryon resp. meson are color-neutral bound states of three quarks resp. quarks and antiquarks. Neutron resp. proton are barions consisting an up- and two down- resp. one down- and two up-quarks. Thus the neutron has no charge while, in the same units in which the electron has an electric charge of -1 , the proton has a charge of $+1$. The total charge of a white particle is always an integer number. Only the quarks confined inside of them can have non-integer charges.

QCD can be regarded as the modern theory of strong nuclear forces holding the quarks together. With no scalar fields, the most general renormalizable Lagrangian describing also these strong interactions can be put in the

form **[W2]**

$$(25) \quad \mathcal{L} = -\frac{1}{4}F^{\alpha\mu\nu}F^\alpha_{\mu\nu} - \sum_n \bar{\psi}_n[x^\mu(\partial_\mu - igA_\mu^\alpha t_\alpha) + m_n]\psi_n,$$

where $\psi_n(x)$ is a matter field, g is the strong coupling constant, t_α are a complete set of generators of color $SU(3)$ in the **3**-representation (that is, Hermitian traceless 3×3 matrices with rows and columns labelled by the three quark colors), normalized so that $Tr(t_\alpha t_\beta) = \delta_{\alpha\beta}/2$, and the subscript n labels quark flavors, with quark color indices suppressed. The first term is called matter Lagrangian density, the second one is the gauge field.

Just as the electromagnetic force between electrons is generated by the virtual exchange of photons, so the quarks are bound to one another by a force that comes from the exchange of other quanta, called gluons because they glue the quarks together to make observable white objects. The gluons are flavor blind, paying no attention to flavor, however, they are very sensitive of color. They interact with color much as the photon interacts with electron charge.

3.2 More specifics about the new abstract model.

The above sketchily described objects are the most fundamental concepts in QCD. Their properties are established by the Lagrangians introduced there. In order to compare them, we review some more details about the new theory. The concepts introduced here will rigorously be establishment in the following sections.

The mathematical structures on which the new theory is built on are 2-step nilpotent Lie groups and their solvable extensions. Both type of manifolds are endowed with natural left invariant metrics. In this scheme, the nilpotent group plays the role of space, on which always positive definite metric is considered. This choice is dictated also by the fact that the Hamilton operators of elementary particle systems emerge as the Laplacians of these invariant Riemann metrics. In order to ensure that these systems have positive energies, just positive definite metric can be chosen on these manifolds; for indefinite metrics the Laplacian never appears as the Hamilton operator of a particle system.

Time can be introduced by adding new dimension to these nilpotent manifolds. This can be implemented either by a simple Cartesian product with the real line \mathbb{R} , or by the solvable extensions of nilpotent groups. Both processes increase the dimension of the nilpotent groups by 1 and in both cases invariant indefinite metrics are defined such that the time-lines intersect the

nilpotent subgroup perpendicularly, furthermore, also $\langle \partial_t, \partial_t \rangle < 0$ holds. On these extended manifolds the Laplacian appears as the natural wave operator (Schrödinger operator) attached to the particle systems. The difference between the two constructions is that the first one provides a static model, while the second one is an expanding model which yields the Hubble law of cosmology.

Although both are relativistic, these space-time concepts are not quite the same than those developed in general relativity. Actually, Einstein's 4D space-time concept has no room for exhibiting the rich "inner life" of particles which is attributed to them by meson theory, or, by the general standard model of elementary particle physics. This "inner life" can not be explained only by the properties of space-time. For instance, the symmetries underlying the electroweak theory are called internal symmetries, because one thinks of them as having to do with the intrinsic nature of the particles, rather than their position or motion.

The abstract mathematical models, however, do make room for both the rich "inner life" and "exterior life" of particles. The main tool for exhibiting the inner physics is the center of the nilpotent Lie algebra, while the stage for the "exterior life" (that is, for the motion of particles) is the so called X-space denoted by \mathcal{X} . This space is a complement of the center \mathcal{Z} , which is called also Z-space.

The space-like Z-space exhibits, actually, dualistic features. The primary meaning of vectors lying in the center is that they are the axes of angular momenta defined for the charged particles which are orbiting in complex planes in constant magnetic fields standing perpendicular to these complex planes. Actually, this axis-interpretation of vectors, Z , is developed in the following more subtle way: For any unit vector Z there is a complex structure, J_Z acting on the X-space corresponded such that the particles are orbiting in the complex planes defined by J_Z along the integral curves of vector fields defined by $X \rightarrow J_Z(X)$. As it is pointed out in the next section, this 2-step nilpotent Lie group is uniquely determined by the linear space, J_Z , of skew endomorphisms J_Z . Bijection $Z \rightarrow J_Z$ provides a natural identification between \mathcal{Z} and J_Z . More precisely, the group can be considered such that it is defined by a linear space of skew angular momentum endomorphisms acting on a Euclidean space, \mathcal{X} , such that it is considered, primarily, as an abstract space \mathcal{Z} which is identified with the endomorphism space, J_Z , by the natural bijection $Z \rightarrow J_Z$. Note that in this interpretation, the axis, Z , of the angular momentum, J_Z , is separated from the complex plane where the actual orbiting is taking place. Anyhow, from this point of view, the Z-vectors exhibit space-like features.

Whereas, the constant magnetic field defined by the structure pins down a unique inertia system on which relations $B = \text{constant}$ and $E = 0$ holds. Thus a naturally defined individualistic inner time is given for each of these particles. This time can be synchronized, allowing to define also a common time, T , which defines the time both on the static models and the solvable extensions. From this point of view, the center exhibits time-like features. This argument clarifies the contradiction between the angular-momentum-axis- and the customary time-axis-interpretation of the center of the Heisenberg groups.

Although the concepts of relativity and quantum theory appear in new forms, they should be considered as refined versions of the original classical objects. Beside the above one, an other example for this claim is the new form by which de Broglie's waves are introduced on these groups. The most important new feature is that the Fourier transform is defined only on the center, $\mathcal{Z} = \mathbb{R}^l$, by the following formula:

$$(26) \quad \int_{\mathbb{R}^l} A(|X|, K) \prod z_i^{p_i}(K_u, X) \bar{z}_i^{q_i}(K_u, X) e^{i(\langle K, Z \rangle - \omega t)} dK,$$

where, for a fixed complex basis \mathbf{B} , complex coordinate system $\{z_i(K_u, X)\}$ on the X -space is defined regarding the complex structure J_{K_u} , for all unit Z -vector K_u . This so called twisted Z -Fourier transform binds the Z -space and the X -space together by the polynomials $\prod z_i^{p_i} \bar{z}_i^{q_i}$ which depend both on the X - and K -variables. It appears also in several other alternative forms. Due to this complexity of the wave functions, the three main forces: the electromagnetic; the weak; and the strong forces of particle theory can be introduced in a unified way such that each of them can be expressed in terms of the weak forces.

The main objects on these abstract structures are the Laplace operators considered both on nilpotent and solvable groups. They turn out to be the Hamilton resp. Schrödinger operators of the particle-systems represented by these metric groups. Due to the fact that these operators are Laplace operators on Riemann manifolds, the conservation of energy is automatically satisfied. In classical quantum theory the Hamilton function, which counts with the total energy of a system, is replaced by the Hamilton operator whose discrete eigenvalues are the quantized energy levels on which the system can exist. On the relativistic mathematical models the total energy is encoded into the Einstein tensor (stress-energy tensor) of the indefinite Riemannian metrics. In the quantum theory developed on these manifold, this stress-energy tensor is replaced by the Laplacians of these manifolds, which are, actually, the Hamilton resp. Schrödinger operators of the particle-systems represented by these models. In other words, this is a

correspondence principle associating the Laplacian resp. the eigenfunction-equations to the Einstein tensor resp. Einstein equation defined on these indefinite Riemann manifolds.

Let it be emphasized again that this theory will be evolved gradually without adding any new objects to those defined mathematically on these abstract structures. The main focus is going to be to rediscover the most important physical features which are known by the standard model. Before starting this exploration, we describe, yet, a more definite bond between the two models.

3.3 Correspondence principle bridging the two models.

Correspondence principle associating 2-step nilpotent groups to $SU(n)$ -models can also be introduced. It can be considered such that, to the Lagrange functions defined on the $SU(n)$ -models, 2-step metric nilpotent Lie groups are corresponded. The combination of this correspondence principle with the above one associates the Laplacian of the metric group to the Lagrange function defined on the Yang-Mills model. This association explains why are the conclusions about the nature of electromagnetic, strong, and weak forces so similar on the two models. This bridge can be built up as follows.

As it is explained above, the invariant Riemann metrics defined on 2-step nilpotent groups, modelling the particle systems in the new theory, can be defined for any linear space, J_Z , of skew endomorphisms acting on the X-space. Thus, for a faithful representation, ρ , of $su(n) \subset so(2n)$ in the Lie algebra of real orthogonal transformations acting on a Euclidean space \mathcal{X} , one can define a natural 2-step nilpotent metric group by the endomorphism space $J_Z = \rho(su(n)) \subset so(\mathcal{X})$ whose X-space is \mathcal{X} and the Z-space is the abstract linear space $Z = \rho(su(n))$. Actually, this is the maximal 2-step nilpotent metric group which can be corresponded to a Yang-Mills principal fibre bundle having structure group $SU(n)$, and whose fibres consists of orthonormal frames of \mathcal{X} on which the action of $\rho(SU(n))$ is one-fold transitive. Note that this group is still independent from the Yang-Mills connection A_μ by which the Yang-Mills field (curvature), $F_{\mu\nu}$, is defined.

By the holonomy group $Hol_p(A) \subset \rho(SU(n))_p$, defined at a fixed point p of the Minkowski space, groups depending on Yang-Mills fields can also be introduced. Even gauge-depending groups can be constructed, by sections $\sigma : M \rightarrow \tilde{M}$, where \tilde{M} denotes the total space of the fibre bundle and $\sigma(p)$ is lying in the fibre over the point p . In this case the endomorphism space is spanned by the skew endomorphisms $A_\mu(\sigma(p))$ considered for all $p \in M$

and indices μ . This construction depends on sections $\sigma(p)$. Since the gauge group is transitive on the set of these sections, this correspondence is not gauge invariant.

These correspondences associate 2-step nilpotent metric groups also to the representations of the particular Lie algebras $su(2) \subset so(4)$ resp. $su(3) \subset so(6)$ by which the Yang-Mills- resp. Gell-Mann~Ne'eman-models are introduced. This association does not mean, however, the equivalence of the two theories. The X-space (exterior world) of the associated group is the Euclidean space \mathcal{X} where the skew endomorphisms from $\rho(su(n))$ are acting. The Z-space is the abstract space $\mathcal{Z} = \rho(g)$ where g can be any of the subspaces of $su(n)$ which were introduced above. In other words, for a Yang-Mills model, X-space \mathcal{X} is already given and angular momentum endomorphism space is picked up in $\rho(su(n))$ in order to define the Z-spaces of the corresponded nilpotent groups. The space-time, including both the exterior and interior worlds, are defined by these spaces. Note that this construction completely ignores the Minkowski space which is the base-space for the Yang-Mills principle fibre bundle. These arguments also show, that the nilpotent groups corresponded to a fixed Yang-Mills model are not uniquely determined, they depend on the Z-space chosen on the YM-model.

Whereas, on a YM-model, the exterior world is the Minkowski space over which the principal fibre bundle is defined and the interior world is defined there by $\rho(su(n))$. An other fundamental difference is that YM-models are based on submersion-theory, however, not this is the case with the new models. For instance, the Hamilton operators of particles never appear as sub-Laplacians defined on the X-space. Quite to the contrary, these Hamilton operators are acting on the total (X, Z) -space, binding the exterior and interior worlds together into an unbroken unity not characteristic for submersions. These are the most important roots explaining both the differences and similarities between the two models. These arguments also show that the new theory deals with much more general situations than those considered in Yang-Mills' resp. Gell-Mann~Ne'eman's $SU(2)$ - resp. $SU(3)$ -theories. It goes far beyond the $SU(n) \times \cdots \times SU(n)$ -theories. Since the Laplacians are natural Hamilton operators of elementary particles also in these most general situations, there is no reason to deny their involvement into particle theory.

Differences arise also regarding the Maxwell theory of electromagnetism. The $SU(n)$ -theories are non-Abelian gauge theories where the field is the curvature, F , of a Yang-Mills connection (potential) and the properties are gauge invariant regarding an infinite dimensional gauge group whose Lie

algebra consists of $su(n)$ -valued 1-forms, ω , satisfying $d\omega = F$. The customary reference to this phenomenon is that only F is the physical object and 1-forms ω do not have any physical significance, they are the results of mere mathematical constructions. This interpretation strongly contrasts the Aharonov-Bohm theory where these vector potentials do have physical meanings by which the effect bearing their names can be established [AB], [T]. Whereas, on the nilpotent groups and their solvable extensions the fundamental fields are the natural invariant Riemann metrics, g , by which all the other objects are defined. The classical Hamilton resp. Schrödinger operators emerge as Laplacians of metrics g . Curvature F corresponds to the field $g(J_Z(X), Y)$ in this interpretation. Since the basic objects are not invariant regarding their actions, the gauge-symmetries are not involved to these investigations. One-form $\omega_Z(Y) = (1/2)g(J_Z(X), Y)$, defined over the points $X \in \mathcal{X}$, is the only element of the Lie algebra of the gauge-symmetry group which is admitted to these considerations. It defines the constant magnetic field and vanishing electric field which are associated with the orbiting spin and inner time T .

This interpretation shows that this model breaks off, at some point, from the Maxwell theory whose greatest achievement was that it unified the, until 1865, separately handled partial theories of magnetism and electricity. The Yang-Mills gauge theory is a generalization of this unified theory to vector valued fields and potentials. The nilpotent Lie group model approaches to the electromagnetic phenomena from a different angle. Since the potentials do have significance there, it stands closer to the Aharonov-Bohm theory than to the Maxwell-Yang-Mills gauge theory. Actually the full recovery of electromagnetism under this new circumstances will be not provided in this paper. Note that only the magnetic field has been appeared so far which is not associated with a non-trivial electric field. In other words, the magnetism and electricity emerge as being separated in this paper. The reunion of this temporarily separated couple can be established just at a later point after further developing this new theory.

This bridge explains the great deal of properties which manifest similarly on these two models, however, turning from one model to the other one is not simple and the complete exploration of overlapping phenomena requires further extended investigations. In this paper the first order task is to firmly establish the point about the new model. Therefore, when several properties are named by the same name used in QCD, their definition strictly refer to our setting. Whereas, by these deliberately chosen names, the similarities between the two theories are indicated.

4 Two-step nilpotent Lie groups.

4.1 Definitions and interpretations.

A 2-step nilpotent metric Lie algebra, $\{\mathcal{N}, \langle, \rangle\}$, is defined on a real vector space endowed with a positive definite inner product. The name indicates that the center, \mathcal{Z} , can be reached by a single application of the Lie bracket, thus its second application always results zero. The orthogonal complement of the center is denoted by \mathcal{X} . Then the Lie bracket operates among these subspaces according to the following formulas:

$$(27) \quad [\mathcal{N}, \mathcal{N}] = \mathcal{Z} \quad , \quad [\mathcal{N}, \mathcal{Z}] = 0 \quad , \quad \mathcal{N} = \mathcal{X} \oplus \mathcal{Z} = \mathbb{R}^k \times \mathbb{R}^l.$$

Spaces \mathcal{Z} and \mathcal{X} are called also Z- and X-space, respectively.

Upto isometric isomorphisms, such a Lie algebra is uniquely determined by the linear space, $J_{\mathcal{Z}}$, of skew endomorphisms $J_Z : \mathcal{X} \rightarrow \mathcal{X}$ defined for any $Z \in \mathcal{Z}$ by the formula

$$(28) \quad \langle [X, Y], Z \rangle = \langle J_Z(X), Y \rangle, \forall Z \in \mathcal{Z}.$$

This statement means that for an orthogonal direct sum, $\mathcal{N} = \mathcal{X} \oplus \mathcal{Z} = \mathbb{R}^k \times \mathbb{R}^l$, of Euclidean spaces a non-degenerated linear map, $\mathbb{J} : \mathcal{Z} \rightarrow SE(\mathcal{X})$, $Z \mapsto J_Z$, from the Z-space into the space of skew endomorphisms acting on the X-space, defines a 2-step nilpotent metric Lie algebra on \mathcal{N} by (28). Furthermore, an other non-degenerated linear map $\tilde{\mathbb{J}}$ having the same range $\tilde{J}_{\mathcal{Z}} = J_{\mathcal{Z}}$ as \mathbb{J} defines isometrically isomorphic Lie algebra.

By means of the exponential map, also the group can be considered such that it is defined on \mathcal{N} . That is, a point is denoted by (X, Z) on the group as well. Then, the group multiplication is given by the formula $(X, Z)(X', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X'])$. Metric tensor, g , is defined by the left invariant extension of \langle, \rangle onto the group N .

Endomorphisms $J_Z(\cdot)$ will be associated with angular momenta. It must be pointed out, however, a major conceptual difference between the classical 3D angular momentum, introduced in (22), and this new sort of angular momentum. For a fixed axis $Z \in \mathbb{R}^3$ the endomorphism associated with the classical 3D angular momentum is defined with the help of the cross product \times by the formula $J_Z : X \rightarrow Z \times X$. That is, axis Z is lying in the same space, \mathbb{R}^3 , where the endomorphism itself is acting. Linear map $\mathbb{J} : \mathcal{Z} \rightarrow SE(\mathcal{X})$ on a 2-step nilpotent Lie group, however, separates axis $Z \in \mathcal{Z}$ from the X-space where the endomorphism $J_Z(\cdot)$ is acting. In other words, the latter endomorphism defines just orbiting of a position vector X

in the plane spanned by X and $J_Z(X)$, but the axis of orbiting is not in the X - but in the Z -space.

In this respect, the Z -space is the abstract space of the axes associated with the angular momentum endomorphisms. According to this interpretation, for a fixed axis Z in the Z -space, a particle occupies a complex plane in the complex space defined by the complex structure J_Z on the X -space. Abstract axis, Z , is considered as an “inner dial” associated with the particles, which is represented separately in the Z -space. This Z -space contributes new dimensions to the X -space which is considered as the inner-world supplemented to the exterior-world in order to have a natural stage for describing the inner physics of elementary particles.

The above definition of 2-step nilpotent Lie groups by their endomorphism spaces J_Z shows the large variety of these groups. For instance, if \mathcal{Z} is an l -dimensional Lie algebra of a compact group and $\mathbb{J} : \mathcal{Z} \rightarrow so(k)$ (which corresponds $J_Z \in so(k)$ to $Z \in \mathcal{Z}$) is its representation in a real orthogonal Lie algebra (that is, in the Lie algebra of skew-symmetric matrices) defined for $\mathcal{X} = \mathbb{R}^k$, then the system $\{\mathcal{N} = \mathcal{X} \oplus \mathcal{Z}, J_Z\}$ defined by orthogonal direct sum determines a unique 2-step nilpotent metric Lie algebra where the inner product on \mathcal{Z} is defined by $\langle Z, V \rangle = -Tr(J_Z \circ J_V)$. Thus, any faithful representation $\mathbb{J} : \mathcal{Z} \rightarrow so(k)$ determines a unique two-step nilpotent metric Lie algebra. Since Lie algebras $su(k/2) \subset so(k)$ used in non-Abelian gauge theories are of compact type, therefore, to any of their representations in orthogonal Lie algebras, one can associate a natural two-step nilpotent metric Lie algebra. This association is the natural bridge between a non-Abelian $SU(n)$ -theory and the new theory developed in this paper.

The 2-step nilpotent Lie groups form even a much larger class than those constructed above by orthogonal Lie algebra representations. In fact, in the most general situation, endomorphism space J_Z is just a linear space defined by the range of a non-degenerated linear map $\mathbb{J} : \mathcal{Z} \rightarrow SE(\mathcal{X})$ which may not bear any kind of Lie algebra structure. However, such general groups can be embedded into those constructed by the compact Lie algebras, $J_{\tilde{\mathcal{Z}}}$. The smallest such Lie algebra is generated by the endomorphism space J_Z by the Lie brackets.

Very important particular 2-step nilpotent Lie groups are the Heisenberg-type groups, introduced by Kaplan [K], which are defined by endomorphism spaces J_Z satisfying the Clifford condition $J_Z^2 = -\mathbf{z}^2 id$, where $\mathbf{z} = |Z|$ denotes the length of the corresponding vector. These groups are attached to Clifford modules (representations of Clifford algebras). The well known

classification of these modules provides classification also for the Heisenberg-type groups. According to this classification, the X-space and the endomorphisms appear in the following form:

$$(29) \quad \mathcal{X} = (\mathbb{R}^{r(l)})^a \times (\mathbb{R}^{r(l)})^b, \quad J_Z = (j_Z \times \cdots \times j_Z) \times (-j_Z \times \cdots \times -j_Z),$$

where $l = \dim(\mathcal{Z})$ and the endomorphisms j_Z act on the corresponding component, $\mathbb{R}^{r(l)}$, of this Cartesian product. The groups and the corresponding natural metrics are denoted by $H_l^{(a,b)}$ and $g_l^{(a,b)}$ respectively. Particularly important examples are the H-type groups $H_3^{(a,b)}$, where the 3-dimensional Z-space, \mathbb{R}^3 , is considered as the space of imaginary quaternions, furthermore, action of j_Z on the space $\mathbb{R}^{r(3)} = \mathbb{H} = \mathbb{R}^4$ of quaternionic numbers is defined by left multiplications with Z .

A brief account on the classification of Heisenberg type groups is as follows. If $l = \dim(J_Z) \not\equiv 3 \pmod{4}$, then, upto equivalence, there exist exactly one irreducible H-type endomorphism space acting on a Euclidean space \mathbb{R}^{n_l} , where the dimensions n_l , which depend just on l , are described below. These endomorphism spaces are denoted by $J_l^{(1)}$. If $l \equiv 3 \pmod{4}$, then, upto equivalence, there exist exactly two non-equivalent irreducible H-type endomorphism spaces acting on \mathbb{R}^{n_l} . They are denoted by $J_l^{(1,0)}$ and $J_l^{(0,1)}$ respectively. They relate to each other by the relation $J_l^{(1,0)} \simeq -J_l^{(0,1)}$.

The values n_l corresponding to $l = 8p, 8p+1, \dots, 8p+7$ are

$$(30) \quad n_l = 2^{4p}, 2^{4p+1}, 2^{4p+2}, 2^{4p+2}, 2^{4p+3}, 2^{4p+3}, 2^{4p+3}, 2^{4p+3}.$$

The reducible Clifford endomorphism spaces can be built up by these irreducible ones. They are denoted by $J_l^{(a)}$ resp. $J_l^{(a,b)}$. The corresponding Lie algebras are denoted by $\mathcal{H}_r^{(a)}$ and $\mathcal{H}_l^{(a,b)}$ respectively, which define the groups $H_r^{(a)}$ resp. $H_l^{(a,b)}$. In the latter case, the X-space is defined by the $(a+b)$ -times product $\mathbb{R}^{n_l} \times \cdots \times \mathbb{R}^{n_l}$ such that, on the last b component, the action of a J_Z is defined by $J_Z^{(0,1)} \simeq -J_Z^{(1,0)}$, and, on the first a components, the action is defined by $J_Z^{(1,0)}$. In the first case this process should be applied only on the corresponding a -times product.

One of the fundamental statements in this theory is that, in case of $l \equiv 3 \pmod{4}$, two groups $H_l^{(a,b)}$ and $H_l^{(a',b')}$ are isometrically isomorphic if and only if $(a,b) = (a',b')$ upto an order. By a general statement, two metric 2-step nilpotent Lie groups with Lie algebras $\mathcal{N} = \mathcal{X} \oplus \mathcal{Z}$ and $\mathcal{N}' = \mathcal{X}' \oplus \mathcal{Z}'$ are isometrically isomorphic if and only if there exist orthogonal transformations

$A : \mathcal{X} \rightarrow \mathcal{X}'$ and $B : \mathcal{Z} \rightarrow \mathcal{Z}'$ such that $J_{B(Z)} = A \circ J_Z \circ A^{-1}$ holds, for all $Z \in \mathcal{Z}$. The isomorphic isometry between $H_l^{(a,b)}$ and $H_l^{(b,a)}$ is defined by $A = id$ and $B = -id$. If $(a, b) \neq (a', b')$ (upto an order) then the corresponding groups are not isometrically isomorphic.

In order to unify the two cases, denotations $J_l^{(1,0)} = J_l^{(1)}$ and $J_l^{(0,1)} = -J_l^{(1)}$ are used also in cases $l \neq 3 \pmod{4}$. One should keep in mind, however, that these endomorphism spaces are equivalent, implying that two groups $H_l^{(a,b)}$ and $H_l^{(a',b')}$ defined by them are isometrically isomorphic if and only if $a + b = a' + b'$ holds.

H-type groups can be characterized as being such particular 2-step metric nilpotent Lie groups on which the skew endomorphisms, J_Z , for any fixed $Z \in \mathcal{Z}$, have the same eigenvalues $\pm \mathbf{zi}$. By polarization we have:

$$(31) \quad \frac{1}{2}(J_{Z_1}J_{Z_2} + J_{Z_2}J_{Z_1}) = -\langle Z_1, Z_2 \rangle Id,$$

which is called Dirac's anticommutation equation. It implies that two endomorphisms, J_{Z_1} and J_{Z_2} , with perpendicular axes, $Z_1 \perp Z_2$, anticommute with each other. Endomorphism spaces satisfying this weaker property define more general, so called totally anticommutative 2-step nilpotent Lie groups on which the endomorphisms can have also properly distinct eigenvalues. The classification of these more general groups is unknown in the literature.

Let it be mentioned, yet, that groups constructed above by $su(2)$ -representations are exactly the groups $H_3^{(a,b)}$, while those constructed by $su(3)$ -representations are not even totally commutative spaces. Thus, they are not H-type groups either. It is also noteworthy, that the Cliffordian endomorphism spaces $J_l^{(a,b)}$ do not form a Lie algebra in general. In fact, only the endomorphism spaces defined for $l = 1, 3, 7$ can form a Lie algebra. In the first two cases, pair (a, b) can be arbitrary, while in case $l = 7$ only cases $a = 1, b = 0$, or, $a = 0, b = 1$ yield Lie algebras. It is an interesting question that which orthogonal Lie algebras can be generated by Cliffordian endomorphism spaces $J_l(a, b)$. Let it be mentioned, yet, that Lie algebra $su(3)$ does not belong even to this category.

4.2 Laplacian and curvature.

Although most of the results of this paper extend to the most general 2-step metric nilpotent Lie groups, in what follows only H-type groups will be

considered. On these groups, the Laplacians appear in the following form:

$$(32) \quad \Delta = \Delta_X + (1 + \frac{1}{4}\mathbf{x}^2)\Delta_Z + \sum_{\alpha=1}^r \partial_\alpha D_\alpha \bullet,$$

where $D_\alpha \bullet$ denotes directional derivatives along the vector fields $X \rightarrow J_\alpha(X) = J_{e_\alpha}(X)$, furthermore, $\mathbf{x} = |X|$ denotes the length of X-vectors.

This formula can be established by the following explicit formulas. Consider orthonormal bases $\{E_1; \dots; E_k\}$ and $\{e_1; \dots; e_l\}$ on the X- and Z-space respectively. The coordinate systems defined by them are denoted by $\{x^1; \dots; x^k\}$ and $\{z^1; \dots; z^l\}$ respectively. Vectors $E_i; e_\alpha$ extend into the left-invariant vector fields

$$(33) \quad \mathbf{X}_i = \partial_i + \frac{1}{2} \sum_{\alpha=1}^l \langle [X, E_i], e_\alpha \rangle \partial_\alpha = \partial_i + \frac{1}{2} \sum_{\alpha=1}^l \langle J_\alpha(X), E_i \rangle \partial_\alpha$$

and $\mathbf{Z}_\alpha = \partial_\alpha$, respectively, where $\partial_i = \partial/\partial x^i$, $\partial_\alpha = \partial/\partial z^\alpha$ and $J_\alpha = J_{e_\alpha}$.

The covariant derivative acts on these invariant vector fields according to the following formulas.

$$(34) \quad \nabla_X X^* = \frac{1}{2} [X, X^*] \quad , \quad \nabla_X Z = \nabla_Z X = -\frac{1}{2} J_Z(X) \quad , \quad \nabla_Z Z^* = 0.$$

The Laplacian, Δ , acting on functions can explicitly be established by substituting (33) and (34) into the following well known formula

$$(35) \quad \Delta = \sum_{i=1}^k (\mathbf{X}_i^2 - \nabla_{\mathbf{X}_i} \mathbf{X}_i) + \sum_{\alpha=1}^l (\mathbf{Z}_\alpha^2 - \nabla_{\mathbf{Z}_\alpha} \mathbf{Z}_\alpha).$$

These formulas allow to compute also the Riemannian curvature, R , on N explicitly. Then we find:

$$(36) \quad R(X, Y)X^* = \frac{1}{2} J_{[X, Y]}(X^*) - \frac{1}{4} J_{[Y, X^*]}(X) + \frac{1}{4} J_{[X, X^*]}(Y);$$

$$(37) \quad R(X, Y)Z = -\frac{1}{4} [X, J_Z(Y)] + \frac{1}{4} [Y, J_Z(X)]; \quad R(Z_1, Z_2)Z_3 = 0;$$

$$(38) \quad R(X, Z)Y = -\frac{1}{4} [X, J_Z(Y)]; \quad R(X, Z)Z^* = -\frac{1}{4} J_Z J_{Z^*}(X);$$

$$(39) \quad R(Z, Z^*)X = -\frac{1}{4} J_{Z^*} J_Z(X) + \frac{1}{4} J_Z J_{Z^*}(X),$$

where $X; X^*; Y \in \mathcal{X}$ and $Z; Z^*; Z_1; Z_2; Z_3 \in \mathcal{Z}$ are considered as the elements of the Lie algebra \mathcal{N} . The components of this tensor field on coordinate systems $\{x^1; \dots; x^k; z^1; \dots; z^l\}$ can be computed by formulas (33).

5 Particles without interior.

These Riemann manifolds were used, originally [Sz1]-[Sz4], for isospectrality constructions in two completely different situations. In the first one, the Z-space is factorized by a Z-lattice, Γ_Z , defined on the Z-space, which process results a torus bundle over the X-space. In the second case, Z-ball resp. Z-sphere bundles are considered by picking Z-balls resp. Z-spheres in the Z-space over the points of the X-space. It turns out that, apart from a constant term, the Laplacian on the Z-torus bundles, called also Z-crystals, is the same as the classical Ginsburg-Landau-Zeeman operator of an electron-positron system whose orbital angular momentum is expressed in terms of the endomorphisms J_{Z_α} defined by the lattice points $Z_\alpha \in \Gamma_Z$. More precisely, the Z-lattice defines a natural decomposition $\sum_\alpha W_\alpha$ of the L^2 -function space such that the components W_α are invariant under the action of the Laplacian, which, after restricting it onto a fixed W_α , appears as the Ginsburg-Landau-Zeeman operator whose orbital angular momentum is associated with the fixed endomorphism J_{Z_α} .

It turns out, in the next chapters, that the constant term corresponds to neutrinos (massless particles with no charge) accompanying an electron-positron system. Thus, altogether, the Laplacian appears as the Hamilton operator of a system of electrons positrons and electron-positron-neutrinos which is formally the sum of a Ginsburg-Landau-Zeeman operator and a constant term. Names Ginsburg-Landau indicate that no Coulomb potential or any kind of electric forces are involved into this operator. Thus the forces which manifest themselves in the eigenfunctions of the Laplacian, are not the complete electromagnetic forces, yet. However, when also these forces will be introduced, the eigenfunctions remain the same, the electric force contributes only to the magnitude of the eigenvalues. By this reason, the forces associated with these models are called electromagnetic forces. Since the Z-lattices consist of points and intrinsic physics is exhibited on the Z-space, particles represented by Z-crystals are considered as point-like particles having no insides. The theory developed for them is in the strongest connection with quantum electrodynamics (QED).

In the second case both the Laplacian and the angular momentum operator appear in much more complex forms. Contrary to the first case, the angular momentum operator is not associated with a fixed Z-vector, but it represents spinning about each Z-vector. Beside the orbiting spin, also natural inner operators emerge which can be associated both with weak and strong nuclear forces. The particles represented by these models do have inside on which the intrinsic physics described in QCD is exhibited on a full scale. An openly admitted purpose in the next sections is to give a unified

theory for the 3 forces: 1.) electromagnetic- 2.) weak-nuclear- 3.) strong-nuclear-forces. However, some concepts such as Dirac's spin operator will be introduced in a subsequent paper into this theory.

5.1 Z-crystals modelling Ginsburg-Landau-Zeeman operators.

The Z-torus bundles are defined by a factorization, $\Gamma \backslash H$, of the nilpotent group H by a Z-lattice, $\Gamma = \{Z_\gamma\}$, which is defined only on \mathcal{Z} and not on the whole (X, Z) -space. Such a factorization defines a Z-torus bundle over the X-space. The natural Z-Fourier decomposition, $L_{\mathbb{C}}^2 := \sum_{\gamma} W_{\gamma}$, of the L^2 function space belonging to this bundle is defined such that subspace W_{γ} is spanned by functions of the form

$$(40) \quad \Psi_{\gamma}(X, Z) = \psi(X) e^{2\pi i \langle Z_{\gamma}, Z \rangle}.$$

Each W_{γ} is invariant under the action of Δ , more precisely we have:

$$(41) \quad \Delta \Psi_{\gamma}(X, Z) = (\triangleleft_{\gamma} \psi)(X) e^{2\pi i \langle Z_{\gamma}, Z \rangle}, \quad \text{where}$$

$$(42) \quad \triangleleft_{\gamma} = \Delta_X + 2\pi i D_{\gamma} \bullet - 4\pi^2 \mathbf{z}_{\gamma}^2 \left(1 + \frac{1}{4} \mathbf{x}^2\right).$$

In terms of parameter $\mu = \pi \mathbf{z}_{\gamma}$, this operator is written in the form $\triangleleft_{\mu} = \Delta_X + 2i D_{\mu} \bullet - \mu^2 \mathbf{x}^2 - 4\mu^2$. Although it is defined in terms of the X-variable, this operator is not a sub-Laplacian resulted by a submersion. It rather is the restriction of the total Laplacian onto the invariant subspace W_{γ} . Actually, the Z-space is represented by the constant μ and operator $D_{\mu} \bullet$. A characteristic feature of this restricted operator is that it involves only a single endomorphism, $J_{Z_{\gamma}}$.

In the 2D-case, such an operator can be transformed to the the Ginsburg-Landau-Zeeman operator (24) by choosing $\mu = eB/2\hbar c$ and multiplying the whole operator with $-\hbar^2/2m$. In general dimensions, number $\kappa = k/2$ means the *number of particles*, and, endomorphisms j_Z and $-j_Z$ in the above formulas are attached to systems electrons resp. positrons. More precisely, by the classification of H-type groups, these endomorphisms are acting on the irreducible subspaces \mathbb{R}^{n_i} and the system is interpreted such that there are $n_i/2$ particles of the same charge orbiting on complex planes determined by the complex structures j_{Z_u} resp. $-j_{Z_u}$ in constant magnetic fields whose directions are perpendicular to the complex planes where the particles are orbiting. The actuality of the complex planes where the orbiting takes place can be determined just probabilistically by the probability amplitudes defined for such systems. The total number of particles is $\kappa = (a+b)n_i/2$. The probability amplitudes must refer to κ number of particles, that is, they are defined on the complex X-space \mathbb{C}^{κ} defined by the complex structure J_{Z_u} .

This theory can be established just after developing an adequate spectral theory.

Above, adjective "perpendicular" is meant to be just symbolic, for axis Z is actually separated from the orbiting. That is, it is not the actual axis of orbiting, but it is a vector in the Z -space which is attributed to the orbiting by the linear map $\mathbb{J} : Z \rightarrow J_Z$. Thus, it would be more appropriate to say that this constant perpendicular magnetic field, B , is just "felt" by the particle orbiting on a given complex plane. Anyhow, the B defines a unique inertia system on the complex plane in which $E = 0$, that is, the associated electric field vanishes. In all of the other inertia system also a non-zero E must be associated with B . The relativistic time T defined on this unique inertia system is the inner time defined for the particle orbiting on a complex plane. This time can be synchronized, meaning that common time T can be introduced which defines time on each complex plane.

Note that this operator contains also an extra constant term, $4\mu^2$, which is explained later as the total energy of neutrinos accompanying the electron-positron system. This energy term is neglected in the original Ginsburg-Landau-Zeeman Hamiltonian. Thus the higher dimensional mathematical model really represents a system of particles and antiparticles which are orbiting in constant magnetic fields. Operator, $D_\mu \bullet$, associated with magnetic dipole resp. angular momentum operators, are defined for the lattice points separately. Therefore this model can be viewed such that it is associated with *magnetic-dipole-moment-crystals*, or, *angular-moment-crystals*. In short, they are called Z -crystals. They are particularly interesting on a group $H_3^{(a,b)}$ where the Z -space is \mathbb{R}^3 . On this Euclidean space all possible crystals are well known by classifications. It would be interesting to know what does this mathematical classification mean from physical point of view and if these Z -crystals really exist in nature?

5.2 Explicit spectra of Ginsburg-Landau-Zeeman operators.

On Z -crystals, the spectral investigation of the total operator (32) can be reduced to the operators \triangleleft_γ , induced by Δ on the invariant subspaces W_γ . On Heisenberg-type groups this operator involves only a single parameter $\mu > 0$, where μ^2 is the single eigenvalue of $-J_\gamma^2$. By this reason, it is denoted by \triangleleft_μ .

This problem is traced back to an ordinary differential operator acting on radial functions, which can be found by seeking the eigenfunctions in the form $F(X) = f(\langle X, X \rangle) \mathbb{H}^{(n,m)}(X)$, where f is an even function defined on

\mathbb{R} and $\mathbb{H}^{(n,m)}(X)$ is a complex valued homogeneous harmonic polynomial of order n , and, simultaneously, it is also an eigenfunction of operator $\mathbf{i}D_\mu \bullet$ with eigenvalue $m\mu$. Such polynomials can be constructed as follows.

Consider a complex orthonormal basis, $\mathbf{B} = \{B_1, \dots, B_\kappa\}$, on the complex space defined by the complex structure $J = (1/\mu)J_\mu$. The corresponding complex coordinate system is denoted by $\{z_1, \dots, z_\kappa\}$. Functions $P = z_1^{p_1} \dots z_\kappa^{p_\kappa} \bar{z}_1^{q_1} \dots \bar{z}_\kappa^{q_\kappa}$ satisfying $p_1 + \dots + p_\kappa = p$, $q_1 + \dots + q_\kappa = n - p$ are n^{th} -order homogeneous polynomials which are eigenfunctions of $\mathbf{i}D_\mu \bullet$ with eigenvalue $m = 2p - n$. However, these polynomials are not harmonic. In order to get the harmonic eigenfunctions, they must be exchanged for the polynomials $\Pi_X^{(n)}(P)$, defined by projections, $\Pi_X^{(n)}$, onto the space of n^{th} -order homogeneous harmonic polynomials of the X -variable. By their explicit description (71), these projections are of the form $\Pi_X^{(n)} = \Delta_X^0 + B_1^{(n)} \mathbf{x}^2 \Delta_X + B_2^{(n)} \mathbf{x}^4 \Delta_X^2 + \dots$, where $\Delta_X^0 = id$. By this formula, also the harmonic polynomial obtained by this projection is an eigenfunction of $\mathbf{i}D_\mu \bullet$ with the same eigenvalue $m\mu$.

When operator \triangleleft_μ is acting on $F(X) = f(\langle X, X \rangle) \mathbb{H}^{(n,m)}(X)$, it defines an ordinary differential operator acting on f . Indeed, by $D_\mu \bullet f = 0$, we have:

$$(43) \quad (\triangleleft_\mu F)(X) = (4\langle X, X \rangle f''(\langle X, X \rangle) + (2k + 4n)f'(\langle X, X \rangle) - (2m\mu + 4\mu^2((1 + \frac{1}{4}\langle X, X \rangle)f(\langle X, X \rangle)))\mathbb{H}^{(n,m)}(X).$$

The eigenvalue problem can, therefore, be reduced to an ordinary differential operator. More precisely, we get:

Theorem 5.1. *On a Z -crystal, $B_R \times T^l$, under a given boundary condition $Af'(R^2) + Bf(R^2) = 0$ defined by constants $A, B \in \mathbb{R}$, the eigenfunctions of \triangleleft_μ can be represented in the form $f(\langle X, X \rangle) \mathbb{H}^{(n,m)}(X)$, where the radial function f is an eigenfunction of the ordinary differential operator*

$$(44) \quad (\triangleleft_{\mu, \tilde{t}} f)(\tilde{t}) = 4\tilde{t}f''(\tilde{t}) + (2k + 4n)f'(\tilde{t}) - (2m\mu + 4\mu^2(1 + \frac{1}{4}\tilde{t}))f(\tilde{t}).$$

For fixed degrees n and m , the multiplicity of such an eigenvalue is the dimension of space formed by the spherical harmonics $\mathbb{H}^{(n,m)}(X)$.

In the non-compact case, when the torus bundle is considered over the whole X -space, the L^2 -spectrum can explicitly be computed. Then, the above functions are sought in the form $f_\mu(\tilde{t}) = u(\tilde{t}\mu)e^{-\tilde{t}\mu/2}$ where $u(\tilde{t})$ is a uniquely determined r^{th} -order polynomial computed for $\mu = 1$. In terms of these parameters, the elements of the spectrum are $\nu_{(\mu, r, n, m)} = -((4r + 4p + k)\mu +$

$4\mu^2$). This spectrum depends just on $p = (m + n)/2$ and the same spectral element appears for distinct degrees n . Therefore, the multiplicity of each eigenvalue is infinity.

Proof. Only the last statement is to be established. We proceed, first, with the assumption $\mu = 1$. Then, function $e^{-\frac{1}{2}\tilde{t}}$ is an eigenfunction of this radial operator with eigenvalue $-(4p + k + 4)$. The general eigenfunctions are sought in the form

$$(45) \quad f(\tilde{t}) = u(\tilde{t})e^{-\frac{1}{2}\tilde{t}}.$$

Such a function is an eigenfunction of \diamond if and only if $u(\tilde{t})$ is an eigenfunction of the differential operator

$$(46) \quad (P_{(\mu=1,n,m)}u)(\tilde{t}) = 4\tilde{t}u''(\tilde{t}) + (2k + 4n - 4\tilde{t})u'(\tilde{t}) - (4p + k + 4)u(\tilde{t}).$$

Because of differentiability conditions, we impose $u'(0) = 0$ on the eigenfunctions. Since in this case, $u(0) \neq 0$ hold for any non-zero eigenfunction, also the condition $u(0) = 1$ is imposed.

In the compact case, corresponding to the ball \times torus-type manifolds defined over a ball B_R , the spectrum of this Laguerre-type differential operator can not be explicitly computed. For a given boundary condition (which can be Dirichlet, $u(R) = 0$, or Neumann, $u'(R) = 0$) the spectrum consists of a real sequence $0 \leq \mu_1 > \mu_2 > \dots \rightarrow -\infty$. The multiplicity of each of these Laguerre-eigenvalues is 1 and the multiplicity corresponding to the Ginsburg-Landau-Zeeman operator is the dimension of the space of spherical harmonics $H^{(n,m)}$. The elements of the Laguerre-spectrum are zeros of a holomorphic function expressed by an integral formula [CoH].

Contrary to the compact case, the spectrum can be explicitly computed for the non-compact torus bundle $\Gamma \backslash H$, defined over the whole X-space. An elementary argument shows that for any $r \in \mathbb{N}$, operator (46) has a uniquely determined polynomial eigenfunction

$$(47) \quad u_{(\mu=1,r,n,m)}(\tilde{t}) = \tilde{t}^r + a_1\tilde{t}^{r-1} + a_2\tilde{t}^{r-2} + \dots + a_{r-1}\tilde{t} + a_r$$

with coefficients satisfying the recursion formulas

$$(48) \quad a_0 = 1 \quad , \quad a_i = -a_{i-1}(r - i)(r + n + \frac{1}{2}k + 1 - i)r^{-1}.$$

Actually, this argument can be avoided and these polynomials can explicitly be established by observing that they are nothing but the Laguerre polynomials defined as the r^{th} -order polynomial eigenfunctions of operator

$$(49) \quad \Lambda_\alpha(u)(\tilde{t}) = \tilde{t}u'' + (\alpha + 1 - \tilde{t})u',$$

with eigenvalues $-r$. This statement follows from identity

$$(50) \quad P_{(\mu=1,n,m)} = 4\Lambda_{(\frac{1}{2}k+n-1)} - (4p + k + 4),$$

implying that the eigenfunctions of operators (46) and (49) are the same and the eigenvalue corresponding to (47) is

$$(51) \quad \nu_{(\mu=1,r,n,m)} = -(4r + 4p + k + 4) \quad , \quad p = \frac{1}{2}(m + n).$$

We also get that, for fixed values of k, n, m (which fix the value also for p), functions $u_{(\mu=1,r,n,m)}$, $n = 0, 1, \dots \infty$ form a basis in $L^2([0, \infty))$.

In case of a single μ , the eigenfunctions are sought in the form

$$(52) \quad u_{\mu r n m}(\langle X, X \rangle) e^{-\frac{1}{2}\mu\langle X, X \rangle} \mathbf{H}^{(n,m)}(X).$$

It turns out that

$$(53) \quad u_{(\mu,r,n,m)}(\tilde{t}) = u_{(\mu=1,r,n,m)}(\mu\tilde{t})$$

and the corresponding eigenvalue is

$$(54) \quad \nu_{(\mu,r,n,m)} = -((4r + 4p + k)\mu + 4\mu^2).$$

This statement can be explained as follows. For a general μ , the action of (55) on a function $f(\tilde{t}) = u(\mu\tilde{t})e^{-\frac{1}{2}\mu\tilde{t}}$ can be described in terms of $\tau = \mu\tilde{t}$ as follows:

$$(55) \quad (L_{(\mu,n,m)}f = \mu(4\tau f_{\tau\tau} + (2k + 4n)f_{\tau} - (2m + \tau)f) - 4\mu^2 f,$$

from which the statement follows. \square

This technique extends to general 2-step nilpotent Lie groups, where the endomorphisms may have distinct eigenvalues, $\{\mu_i\}$. In this case the eigenfunctions are represented as products of functions of the form

$$(56) \quad F_{(i)}(X) = f_{(i)}(\langle X, X \rangle) \mathbf{H}^{(n_i, m_i)}(X),$$

where the functions in the formula are defined on the maximal eigensubspace corresponding to the parameter μ_i . When the spectrum is computed on the whole X-space, this method works out for the most general Ginsburg-Landau-Zeeman operators. In case of a single μ , this method applies also to computing the spectra on torus bundles over balls and spheres. In case of multiple μ 's, it applies to torus bundles over the Cartesian product of balls resp. spheres defined on the above X_i -spaces.

6 Particles having interior.

In order to sketch up a clear map for this rather complex section, we start with a review of the main mathematical and physical ideas this exposition is based on. These ideas are rigorously established in the subsequent subsections.

6.1 A preliminary review of the main ideas.

Systems of particles having insides can be attached to ball \times ball- and ball \times -sphere-type manifolds. Originally they emerged in the second type of spectral investigations performed in [Sz2]-[Sz4]. These manifolds are defined by appropriate smooth fields of Z-balls resp. Z-spheres of radius $R_Z(\mathbf{x})$ over the points of a fixed X-ball B_X whose radius is denoted by R_X . Note that radius $R_Z(\mathbf{x})$ depends just on the length, $\mathbf{x} := |X|$, of vector $X \in B_X$ over which the Z-balls resp. Z-spheres are considered. The centers all of the balls resp. spheres which show up in this definition are always at the origin of the corresponding spaces. The boundaries of these manifolds are the so called sphere \times ball- resp. sphere \times sphere-type manifolds, which are trivial Z-ball- resp. Z-sphere-bundles defined over fixed X-spheres of radius R_X . In short, one considers Z-balls resp. Z-spheres instead of the Z-tori used in the previous constructions of Z-crystals. In the isospectrality investigations these compact domains corresponding to $R_X < \infty$ play the primary interest. In physics, however, the non-compact bundles corresponding to $R_X = \infty$ (that is, which are defined over the whole X-space) become the most important cases. In what follows, both the compact and non-compact cases will be investigated.

Contrary to the Z-crystal models, the computations in this case can not be reduced to a single endomorphism. Instead, they always have to be established for the complete operator $\mathbf{M} = \sum \partial_\alpha D_\alpha \bullet$ which includes the angular momentum endomorphisms J_Z with respect to any Z-directions. This operator strongly relates both to the 3D angular momentum $\mathbf{P} = (P_1, P_2, P_3) = \frac{1}{\hbar} \mathbf{Z} \times \mathbf{p}$, defined in (21), and the strong interaction term $\mathbf{i} g x^\mu A_\mu^\alpha t_\alpha$ of the QCD Lagrangian (25). Actually, it has a rather apparent formal identity with Pauli's intrinsic spin Hamiltonian $\sum B_i P_i$ defined by magnetic fields $\mathbf{B} = (B_1, B_2, B_3)$ (cf. [P], Volume 5, pages 152-159). In $SU(3)$ -theory the term corresponding to the 3D angular momentum is exactly the above mentioned strong interaction term. Due to the new form (26) of the de Broglie waves, where the angular-momentum-axes are separated from the planes on which the particles are orbiting, also the angular momentum operator has to appear in a new form.

In order to make the argument about the analogy with Pauli's intrinsic spin resp. strong interaction term more clear, note that the Lie algebras determined by the 3D angular momenta resp. Gell-Mann's matrices t_α are $su(2)$ resp. $su(3)$. Thus, the nilpotent-group-models corresponding to these classical Yang-Mills models are $H_3^{(a,b)}$ resp. the nilpotent group constructed by the representations of $su(3)$. When operator \mathbf{M} acts on wave function (26), it appears behind the integral in the form $\mathbf{i}D_K\bullet$. Consider, first, group $H_3^{(a,b)}$ and suppose that $K = e_1$, where $\{e_1, e_2, e_3\}$ is the natural basis on \mathbb{R}^3 . Then, on the $\{e_2, e_3\}$ -plane, which is a complex plane regarding the complex structure J_{e_1} , operator $\mathbf{i}D_{e_1}\bullet$ is nothing but the first component, $-P_1 = \mathbf{i}(Z_2\frac{\partial}{\partial Z_3} - Z_3\frac{\partial}{\partial Z_2})$, of Pauli's angular momentum operator. That is, the analogy between \mathbf{M} and the classical angular momentum operator becomes apparent after letting \mathbf{M} act on wave functions (26). This action is the very same how P is acting on the original de Broglie waves. Thus this new form of action, which can be described by axis-separation and placing the orbiting particles onto the complex planes, can really be considered as adjustment to the new forms of the wave functions. These arguments work out also for groups constructed by $su(3)$ -representations. Thus it is really justified to consider \mathbf{M} as a spin operator appearing in a new situation. The greatest advantage of this new form is that it describes also the strong nuclear forces.

This complication gives rise to a much more complex mathematical and physical situation where both the exterior and the interior life of particle systems exhibit themselves on a full scale. First, let the physical role of the Fourier transforms appearing in the formulas be clarified. If term involving time is omitted from the formula of wave functions, the rest is called time-less probability amplitude. Both these amplitudes and wave functions could have been defined also by means of the inverse function $e^{-\mathbf{i}\langle Z, K \rangle}$. Wave functions (or amplitudes) obtained from the very same function by using $e^{\mathbf{i}\langle Z, K \rangle}$ resp. $e^{-\mathbf{i}\langle Z, K \rangle}$ in their Fourier transforms are said to be wave functions (or amplitudes) defined for particle- resp. antiparticle-systems. In other words, the definition of particles and antiparticles is possible because of these two choices. Calling one object particle and its counter part antiparticle is very similar to naming one of the poles of a magnet north-pole and the other one south pole. Since the Laplace operators on 2-step nilpotent Lie groups are defined by means of constant magnetic fields, this is actually the right physical explanation for choosing $e^{\mathbf{i}\langle Z, K \rangle}$ or $e^{-\mathbf{i}\langle Z, K \rangle}$ to introduce probability amplitudes. By this reason, \mathbf{M} is called unpolarized magnetic dipole moment or angular momentum operator. The polarized operators appear behind the integral sign of the Fourier integral formula when \mathbf{M} is acting on the formula.

The complexity of this operator is fascinating. For instance, it is the sum of extrinsic, \mathbf{L} , and intrinsic, \mathbf{S} , operators which do not commute with each other. Furthermore, operator $\mathbb{E} = \Delta_X + (1 + \frac{1}{4}\mathbf{x}^2)\Delta_Z + \mathbf{L}$ is a Ginsburg-Landau-Zeeman operator which exhibits just orbital spin. The intrinsic life of particles is encoded into \mathbf{S} . Also the strong nuclear forces, keeping the particles having interior together, can be explained by this operator. In order to make this complicated situation as clear as possible, we describe, in advance, how certain eigenfunctions of Δ can explicitly be computed. These computations provide a great opportunity also for a preliminary review of the general eigenfunction computations, which will be connected to these particular computations as follows.

Since these particular functions do not satisfy the boundary conditions, they do not provide the final solutions for finding the eigenfunctions yielding also given boundary conditions. This conditions can be imposed just after certain projections performed on the center (that is, in the insides of the particles). But then, these projected functions will not be eigenfunctions of the complete Δ any more. They are eigenfunctions just of the exterior operator \mathbb{E} . By this reason, they are called weak force eigenfunctions. The strong force eigenfunctions, defined by the eigenfunctions of the complete Δ satisfying also a given boundary condition, can be expressed just by complicated combinations of the weak force eigenfunctions, meaning that the strong forces are piled up by weak forces. Since the weak force eigenfunctions are Ginsburg-Landau-Zeeman eigenfunctions, by the combinations of which the strong force eigenfunctions can be expressed, this theory really unifies the electromagnetic, the weak, and the strong nuclear forces. Let it be mentioned yet that the only force-category missing from this list is the gravitational force. At this early point of the development, we do not comment the question if this unification can be extended also to this force.

Now we turn back to establish an explicit formula describing certain eigenfunctions of Δ on general Heisenberg type Lie groups $H_l^{(a,b)}$. More details about the general eigenfunction computations will also be provided. Although this construction can be implemented also on general 2-step nilpotent Lie groups, this more complicated case is omitted in this paper. First, the eigenfunctions of a single angular momentum operator $\mathbf{D}_{K\bullet}$, defined for a Z -vector K are described as follows. For a fixed X -vector Q and unit Z -vector $K_u = \frac{1}{\mathbf{k}}K_u$, consider the X -function $\Theta_Q(X, K_u) = \langle Q + \mathbf{i}J_{K_u}(Q), X \rangle$ and its conjugate $\overline{\Theta}_Q(X, K_u)$. For a vector $K = \mathbf{k}K_u$ of length \mathbf{k} , these functions are eigenfunctions of $D_{K\bullet}$ with eigenvalue $-\mathbf{k}\mathbf{i}$ resp. $\mathbf{k}\mathbf{i}$. The higher order eigenfunctions are of the form $\Theta_Q^p \overline{\Theta}_Q^q$ with eigenvalue $(q - p)\mathbf{k}\mathbf{i}$.

In order to find the eigenfunctions of the compound operator \mathbf{M}_Z , consider a Z-sphere bundle $S_{R_Z}(\mathbf{x})$ over the X-space whose radius function $R_Z(\mathbf{x})$ depends just on $|X| = \mathbf{x}$. For an appropriate function $\phi(\mathbf{x}, K)$ (depending on \mathbf{x} and $K \in S_{R_Z}$, furthermore, which makes the following integral formula well defined) consider

$$(57) \quad \mathcal{F}_{QpqR_Z}(\phi)(X, Z) = \oint_{S_{R_Z}} e^{i\langle Z, K \rangle} \phi(\mathbf{x}, K) (\Theta_Q^p \bar{\Theta}_Q^q)(X, K_u) dK_{no},$$

where dK_{no} is the normalized measure on $S_{R_Z}(\mathbf{x})$. By $\mathbf{M}_Z \mathcal{f} = \mathcal{f} \mathbf{i} D_K \bullet$, this function restricted to the Z-space over an arbitrarily fixed X-vector is an eigenfunction of \mathbf{M}_Z with the real eigenvalue $(p - q)R_Z(\mathbf{x})$. These functions are eigenfunctions also of Δ_Z with eigenvalue $R_Z^2(\mathbf{x})$. Also note that these eigenvalues do not change by varying Q .

The function space spanned by functions (57) which are defined by all possible ϕ 's is not invariant with respect to the action of Δ_X , thus the eigenfunctions of the complete operator Δ do not appear in this form. In order to find the common eigenfunctions, the homogeneous but non-harmonic polynomials $\Theta_Q^p \bar{\Theta}_Q^q$ of the X-variable should be exchanged for the polynomials $\Pi_X^{(n)}(\Theta_Q^p \bar{\Theta}_Q^q)$, defined by projections, Π_X , onto the space of $n = (p + q)$ -order homogeneous harmonic polynomials of the X-variable. Formula $\Pi_X = \Delta_X^0 + B_1 |X|^2 \Delta_X + B_2 \mathbf{x}^4 \Delta_X^2 + \dots$, established in (71), implies that, over each X-vector, also

$$(58) \quad \mathcal{H}\mathcal{F}_{QpqR_Z}(\phi)(X, Z) = \oint_{S_{R_Z}} e^{i\langle Z, K \rangle} \phi(\mathbf{x}, K) \Pi_X^{(n)}(\Theta_Q^p \bar{\Theta}_Q^q)(X, K_u) dK_{no}$$

are eigenfunctions of \mathbf{M} and Δ_Z with the same eigenvalues what are defined for (57).

The action of the complete Laplacian is a combination of X-radial differentiation, $\partial_{\mathbf{x}}$, and multiplications with functions depending just on \mathbf{x} . Due to the normalized measure dK_{no} , these operations can be considered such that they directly act inside of the integral sign on function $\phi(\mathbf{x}, K)$ in terms of the \mathbf{x} -variable, only. That is, the action is completely reduced to X-radial functions and the eigenfunctions of Δ can be found in the form

$$(59) \quad f(\mathbf{x}^2) \oint_{S_{R_Z}} e^{i\langle Z, K \rangle} F_{Qpq}(X, K_u) dK_{no}, \quad \text{where}$$

$$(60) \quad F_{Qpq}(X, K_u) = \varphi(K) \Pi_X^{(n)}(\Theta_Q^p(X, K_u) \bar{\Theta}_Q^q(X, K_u)).$$

The same computations developed for the Z-crystals yield that this reduced operator appears in the following form:

$$(61) \quad (\Diamond_{\mu(\tilde{t}), \tilde{t}} f)(\tilde{t}) = 4\tilde{t} f''(\tilde{t}) + (2k + 4n) f'(\tilde{t}) - (2m\mu(\tilde{t}) + 4\mu^2(\tilde{t})(1 + \frac{1}{4}\tilde{t})) f(\tilde{t}),$$

where $\tilde{t} = \mathbf{x}^2$, and $\mu(\tilde{t}) = R_Z(\sqrt{\tilde{t}}) = R_Z(\mathbf{x})$. Note that this is exactly the same operator what was obtained for the radial Ginsburg-Landau-Zeeman operator on Z-crystals. Since function $\mu(\tilde{t})$ may depend also on $\tilde{t} = \mathbf{x}^2$, it actually appears in a more general form here. However, for constant radius functions R_Z , it becomes the very same operator, indeed. That is, also this eigenfunction-problem is reduced to finding the eigenfunctions of the Ginsburg-Landau-Zeeman operator reduced to X-radial functions. This reduced operator remains the same by varying Q , thus also the spectrum on the invariant spaces considered for fixed Q 's is not changing regarding these variations. This phenomena reveals the later discussed spectral isotropy yielded on these models.

Note that this construction is carried out by a fixed X-vector Q , but it extends to general polynomials as follows. Consider an orthonormal system $\mathbf{B} = \{B_1, \dots, B_\kappa\}$ of vectors on the X-space. They form a complex, but generically non-orthonormal basis regarding the complex structures J_{K_u} , where the unit vectors K_u yielding this property form an everywhere dense open set on the unit Z-sphere. This set is the complement of a set of 0 measure. The corresponding complex coordinate systems on the X-space are denoted by $\{z_{K_u 1} = \Theta_{B_{K_u 1}}, \dots, z_{K_u \kappa} = \Theta_{B_{K_u \kappa}}\}$. For given values $p_1, q_1, \dots, p_\kappa, q_\kappa$, consider the polynomial $\prod_{i=1}^\kappa z_{K_u i}^{p_i} \bar{z}_{K_u i}^{q_i}$. Then functions

$$(62) \quad \oint_{S_{R_Z}} e^{i\langle Z, K \rangle} f(\mathbf{x}^2) \varphi(K) \Pi_X^{(n)} \prod_{i=1}^\kappa z_{K_u i}^{p_i} \bar{z}_{K_u i}^{q_i} dK_{no} = \\ = f(\mathbf{x}^2) \oint_{S_{R_Z}} e^{i\langle Z, K \rangle} F_{\mathbf{B}_{p_i q_i}}(X, K_u) dK_{no} = \mathcal{H}\mathcal{F}_{\mathbf{B}_{p_i q_i} R_Z}(f)(X, Z)$$

are eigenfunctions of Δ if and only if function $f(\mathbf{x}^2) = f(\tilde{t})$ is an eigenfunction of the radial operator (61), where $p = \sum p_i, q = \sum q_i, n = p + q$.

Consider a Z-ball bundle with radius function $R_Z(\mathbf{x})$ defining a compact or non-compact ball \times ball-type domain. Then the eigenfunctions satisfying the Dirichlet or Z-Neumann conditions on this domain can not be sought among the above eigenfunctions because functions $F_{Qpq}(X, K_u)$ resp. $F_{\mathbf{B}_{p_i q_i}}(X, K_u)$ are not spherical harmonics regarding the K_u -variable but they are rather combinations of several spherical harmonics belonging to different eigenvalues of the Z-spherical Laplacian. It turns out, however, that one can construct the complete function space satisfying a given boundary condition by the above formulas if functions F_{\dots} are substituted by their projections $\Pi_{K_u}^{(s)}(F_{\dots})$ into the space of s^{th} -order spherical harmonics regarding variable K_u . Actually, this projection appears in a more subtle form, $\Pi_{K_u}^{(vas)} = \Pi_{K_u}^{(\alpha)}$, which projects F_{\dots} , first, into the space of $(v+a)^{th}$ -order homogeneous polynomials (where v resp. a refer to the degrees of functions to which $\varphi(K)$

resp. the $(p + q)^{th}$ -order polynomials are projected). The projection to the s^{th} -order function space applies, then, to these homogeneous functions. The most important mathematical tool applied in this investigations is the Hankel transform developed later.

Although these new functions yield the boundary conditions, they do not remain eigenfunctions of the complete Laplacian anymore. However, they are still eigenfunctions of the partial operator \mathbb{E} . They are called weak-force-eigenfunctions, which can be considered as electromagnetic-force-eigenfunctions because both \mathbb{E} and the Ginsburg-Landau-Zeeman operators can be reduced to the same radial operator. The old ones from which the new functions are derived are called linkage-eigenfunctions of Δ , which bridge the electromagnetic interactions with the weak interactions.

Since the extrinsic, \mathbf{L} , and intrinsic, \mathbf{S} , operators do not commute, the weak-force-eigenfunctions can not be the eigenfunctions of the complete operator $\Delta = \mathbb{E} + \mathbf{S}$. In other words, the weak-force-eigenfunctions can not be equal to the strong-force-eigenfunctions defined by the eigenfunctions of Δ satisfying a given boundary condition. To construct these functions, the Fourier integrals must be considered on the whole \mathbf{Z} -space \mathbb{R}^l , by seeking them in the form $\int_{\mathbb{R}^l} e^{i\langle \mathbf{Z}, \mathbf{K} \rangle} \phi_\alpha(\mathbf{x}, \mathbf{z}) \Pi_K^{(\alpha)}(F_{\dots}(X, K)) dK$. The action of Δ on these functions can be described in the form $\int_{\mathbb{R}^l} e^{i\langle \mathbf{Z}, \mathbf{K} \rangle} \bigcirc_\alpha(\phi_1, \dots, \phi_d) \Pi_K^{(\alpha)}(F_{\dots}(X, K)) dK$, where operator $\bigcirc_\alpha(\phi_1, \dots, \phi_d)$, corresponding d-tuples of (X, Z) -radial functions to each other, is defined in terms of Hankel transforms combined with radial derivatives of the functions appearing in the arguments. Finding the eigenfunctions of Δ means finding the eigen-d-tuples, (ϕ_1, \dots, ϕ_d) , of the radial, so called roulette operator \bigcirc_α . As it will be pointed out, these eigenfunctions exhibit properties characteristic to strong force eigenfunctions.

These arguments really unify the 3 fundamental forces of particle theory. The real union is exhibited by the common unpolarized operator Δ . After polarization, they are separated into three categories. These cases correspond to the function spaces on which the polarized operators are acting. The details are as follows.

6.2 Twisted Z-Fourier transforms.

This is the main mathematical tool which incorporates de Broglie's wave theory into the new models in a novel, more general form. The name indicates that the Fourier transform is performed, over each \mathbf{X} -vector, only on the \mathbf{Z} -space in the same manner as if one would like to consider the de

Broglie waves only in the center of the Lie group. However, an important new feature is that this transform applies to product of functions, where one of them purely depends just on the center variable, K , while the other is a complex polynomial of the X -variable defined in terms of the complex structures J_{K_u} , where $K_u = K/\mathbf{k}$ and $\mathbf{k} = |K|$. This Z-Fourier transform is said to be twisted by the latter polynomials. Thus this transform has impact also on the X -variable. This simple idea establishes the necessary connection between the abstract axes, K_u , and the particles placed onto the complex planes of the complex structures J_{K_u} .

This transform is defined in several alternative forms corresponding to those introduced in the previous review. The difference is that, over each X -vector, the following functions and integrals are defined on the whole Z -space. This is contrary to the previous section where the integral is defined just on Z -spheres. Since the eigenfunctions constructed in the review do not satisfy any of the boundary conditions, this reformulation of the Z-Fourier transform is really necessary for the complete solutions of the considered problems.

In the first case, consider a fixed X -vector Q , and define the same functions

$$(63) \quad \Theta_Q(X, K_u) = \langle Q + \mathbf{i}J_{K_u}(Q), X \rangle, \quad \bar{\Theta}_Q(X, K_u),$$

as above. For fixed integers $p, q \geq 0$ and L^2 -function $\phi(\mathbf{x}, K)$, consider the Z-Fourier transform

$$(64) \quad \mathcal{F}_{Qpq}(\phi)(X, Z) = \int_{\mathbf{z}} e^{\mathbf{i}\langle Z, K \rangle} \phi(\mathbf{x}, K) \Theta_Q^p(X, K_u) \bar{\Theta}_Q^q(X, K_u) dK,$$

which is said to be twisted by the polynomial $\Theta_Q^p(X, K_u) \bar{\Theta}_Q^q(X, K_u)$. Function ϕ is considered also in the form $\phi(\mathbf{x}, \mathbf{k})\varphi(K_u)$, where φ is a homogeneous polynomial of the K -variable. The L^2 -space spanned by these functions is denoted by $\Phi_{Qpq}^{(n)}$, where $n = p+q$ indicates that these functions are n^{th} -order polynomials regarding the X -variable. The space spanned by the twisted functions $\phi(\mathbf{x}, K) \Theta_Q^p(X, K_u) \bar{\Theta}_Q^q(X, K_u)$ is denoted by $\mathbf{P}\Phi_{Qpq}^n$. This is the pre-space to which the Fourier transform is applied.

Instead of a single vector Q , the second alternative form is defined regarding $\kappa = k/2$ independent vectors, $\mathbf{B} = \{E_1, \dots, E_\kappa\}$, of the X -space. Such a system forms a complex basis for almost all complex structure J_{K_u} , where $K_u = K/\mathbf{k}$. Now the twisting functions are polynomials of the complex coordinate functions

$$(65) \quad \{z_{K_u 1}(X) = \Theta_{Q_1}(X, K_u), \dots, z_{K_u \kappa}(X) = \Theta_{Q_\kappa}(X, K_u)\},$$

where these formulas indicate that how the coordinate functions can be expressed in terms of the above Θ_Q -functions. For appropriate functions

$\phi(\mathbf{x}, K)$ and polynomial exponents (p_i, q_i) (where $i = 1, \dots, k/2$) transform $\mathcal{F}_{\mathbf{B}(p_i q_i)}(\phi\varphi)(X, Z)$ is defined by:

$$(66) \quad \int_{\mathbb{R}^l} e^{i\langle Z, K \rangle} \phi(\mathbf{x}, \mathbf{k}) \varphi(K_u) \prod_{i=1}^{\kappa} z_{K_{u_i}}^{p_i}(X) \bar{z}_{K_{u_i}}^{q_i}(X) dK,$$

where $\varphi(K_u)$ is the restriction of an m^{th} -order homogeneous polynomial, $\varphi(K)$, onto the unit sphere of the K-space. If $\sum_i (p_i + q_i) = n$, then the twisting functions are n^{th} -order complex valued polynomials regarding the X-variable and for any fixed X , the whole function is of class L_K^2 regarding the K-variable. These properties are inherited also for the transformed functions.

When $\phi(\mathbf{x}, K)$ runs through all functions which are of class L_K^2 , for any fixed $|X|$, the transformed functions span the function space $\Phi_{\mathbf{B} p_i q_i}^{(n)}$. All these function spaces, defined for index sets satisfying $n = \sum (p_i + q_i)$, span the function space denoted by $\Phi_{\mathbf{B}}^{(n)} = \sum_{\{(p_i q_i)\}} \Phi_{\mathbf{B} p_i q_i}^{(n)}$. The corresponding pre-spaces are denoted by $\mathbf{P}\Phi_{\mathbf{B} p_i q_i}^{(n)}$ and $\mathbf{P}\Phi_{\mathbf{B}}^{(n)}$ respectively.

Twisted Z-Fourier transforms (57)-(62) defined in the previous section by considering Dirac-type functions concentrated on spheres S_R of radius $R(\mathbf{x})$ can be generated by the familiar L^∞ approximation of these Dirac type functions by L^2 functions. That is, this radius depends just on \mathbf{x} . Function spaces $\Phi_{\mathbf{B} p_i q_i R}^{(n)}$, $\Phi_{\mathbf{B} R}^{(n)}$, and their pre-spaces $\mathbf{P}\Phi_{\mathbf{B} p_i q_i R}^{(n)}$ resp. $\mathbf{P}\Phi_{\mathbf{B} R}^{(n)}$ are defined similarly as for L^2 functions. They are defined also for one-pole functions. These cases are denoted such that \mathbf{B} is replaced by Q .

In many cases the theorems hold true for each version of these function spaces. By this reason we introduce the unified denotation $\Phi_{\dots R}^{(n)}$ and $\mathbf{P}\Phi_{\dots R}^{(n)}$, where the dots represent the symbols introduced above on the indicated places. A unified denotation for the total spaces are $\Phi_{\cdot R}^{(n)}$ and $\mathbf{P}\Phi_{\cdot R}^{(n)}$, indicating that symbols $p_i q_i$ do not show up in these formulas. If letter R is omitted, the formulas concern the previous cases when the functions are of class L^2 regarding the K -variable.

The elements of the latter total function spaces are complex valued functions defined on the (X, Z) -space such that, upto multiplicative X-radial functions, they are n^{th} -order polynomials regarding the X-variable and for any fixed X they are L^2 functions regarding the Z-variable. It is very important to clarify the relations between the above twisted spaces and the latter complex valued function spaces. By considering an arbitrary real

basis $\mathbf{Q} = \{Q_1, \dots, Q_k\}$ on the X-space, the complex valued functions appear in the form $\sum_{\{a_1, \dots, a_k\}} \phi_{a_1, \dots, a_k}(|X|, Z) \prod_{i=1}^k \langle Q_i, X \rangle^{a_i}$, where the sum is considered for all sets $\{a_1, \dots, a_k\}$ of non-negative integers satisfying $\sum a_i = n$ and functions ϕ are L_Z^2 -functions for any fixed $|X|$. The function space spanned by these functions is called *straight L_Z^2 space of complex valued (X, Z) -functions*. The Z-Fourier transform performed on such functions is called *straight Z-Fourier transform*. The explanations below show that twisted functions $\phi(|X|, K) \prod_{i=1}^{k/2} z_{K_u i}^{p_i}(X) \bar{z}_{K_u i}^{q_i}(X)$ can be converted to straightly represented functions in the terms of which the twisted Z-Fourier transform becomes a straight Z-Fourier transform. This conversion works out also in the opposite (from the straight to the twisted) direction. The precise details below show that the straightly defined function spaces are complete regarding the L_Z^2 norm in which both $\Phi_{\mathbf{B}}^n$ and the pre-space $\mathbf{P}\Phi_{\mathbf{B}}^n$ are everywhere dense subspaces. By this reason, the corresponding straight spaces are denoted by $\bar{\Phi}_{\mathbf{B}}^n$ resp. $\overline{\mathbf{P}\Phi}_{\mathbf{B}}^n$. However, these spaces are equal. Ultimately, both versions of the Z-Fourier transforms define automorphisms (one to one and onto maps) of this very same ambient function space.

This situation can be illuminated by the real $k \times k$ matrix field $A_{ij}(K_u)$ defined on the unit Z-vectors which transforms the real basis \mathbf{Q} to the vector system $\mathbf{B}_{\mathbb{R}} = \{B_1, \dots, B_{k/2}, B_{(k/2)+1} = J_{K_u}(B_1), \dots, B_k = J_{K_u}(B_{k/2})\}$. That is, this field is uniquely determined by the formula $B_i = \sum_{j=1}^k A_{ij} Q_j$, where $i = 1, \dots, k$. The entries are polynomials of K_u . By plugging these formulas into the twisted Z-Fourier transform formula, one gets the straight representations both of the twisted functions and their Z-Fourier transforms.

Conversion from the straight to the twisted functions is more complicated. In this case vectors Q_i should be exchanged for vectors B_j according to the formula $Q_i = \sum_{j=1}^k A_{ij}^{-1} B_j$. Then vectors B_j resp. $B_{(k/2)+j}$, where $j \leq k/2$, should be expressed in the form

$$(67) \quad B_j = \frac{1}{2}(B_j + J_{K_u}(B_j)) + \frac{1}{2}(B_j - J_{K_u}(B_j)) \quad \text{resp.}$$

$$(68) \quad B_{(k/2)+j} = J_{K_u}(B_j) = \frac{1}{2}(B_j + J_{K_u}(B_j)) - \frac{1}{2}(B_j - J_{K_u}(B_j)),$$

which, after performing powering and appropriate rearranging, provide the desired twisted formulas. Due to the degeneracy of the matrix field A_{ij} on $S_{\mathbf{B}}$, some entries of the inverse matrix field A_{ij}^{-1} have limits $+\infty$ or $-\infty$ of order at most $k/2$ on this singularity set. Therefore, the Z-Fourier transform of a term of the twisted function involving such functions may be not defined, despite the fact that the Z-Fourier transform of the whole function exist which is equal to the transform of the straightly represented function. In

other words, the infinities appearing in the separate terms cancel each other out in the complete function.

This contradictory situation can be resolved as follows. For a given $\epsilon > 0$, let $S_{\mathbf{B}\epsilon}$ be the ϵ -neighborhood of the singularity set on the unit \mathbf{Z} -sphere and let $\mathbb{R}S_{\mathbf{B}\epsilon}$ be the conic set covered by the rays emanating from the origin which are spanned by unit \mathbf{Z} -vectors pointing to the points of $S_{\mathbf{B}\epsilon}$. For an $L^2_{\mathbf{Z}}$ -function $\phi(\mathbf{x}, K)$ discussed above let $\phi_\epsilon(\mathbf{x}, K)$ be the function which is the same as ϕ on the outside of $\mathbb{R}S_{\mathbf{B}\epsilon}$ and it is equal to zero in the inside of this set. Then, regarding the $L^2_{\mathbf{Z}}$ -norm, $\lim_{\epsilon \rightarrow 0} \phi_\epsilon = \phi$ holds. For a function $F(X, K)$, expressed straightly, define $F_\epsilon(X, K)$ by substituting each function ϕ by ϕ_ϵ . If function $\psi_\epsilon(K)$ is defined by 1 outside of $\mathbb{R}S_{\mathbf{B}\epsilon}$ and by 0 inside, then $F_\epsilon(X, K) = \psi_\epsilon(K)F(X, K)$ holds. Convert F_ϵ into the twisted form. Then the twisted \mathbf{Z} -Fourier transform is well defined for each term of the twisted expression, providing the same transformed function as what is defined by the straight \mathbf{Z} -Fourier transform. Thus each straightly represented $L^2_{\mathbf{Z}}$ function is an $L^2_{\mathbf{Z}}$ -limit of functions which can be converted to twisted functions in which each twisted term is an $L^2_{\mathbf{Z}}$ function having well defined twisted \mathbf{Z} -Fourier transform. In this process, function $\psi_\epsilon(K)$, which is constant in radial directions, can be chosen such that it is of class C^∞ satisfying $0 \leq \psi_\epsilon(K) \leq 1$, furthermore, it vanishes on $\mathbb{R}S_{\mathbf{B}\epsilon/2}$ and is equal to 1 outside of $\mathbb{R}S_{\mathbf{B}\epsilon}$. Thus we have:

Theorem 6.1. *The twisted functions, defined in each of their terms by $L^2_{\mathbf{Z}}$ functions, form an everywhere dense subspace $\mathbf{P}\Phi_{\mathbf{B}}^n$ in the complete space $\overline{\mathbf{P}\Phi_{\mathbf{B}}}^n$ of straightly defined $L^2_{\mathbf{Z}}$ functions. Although the first space depends on \mathbf{B} , the second one is a uniquely determined function space which does not depend neither on \mathbf{B} nor on \mathbf{Q} .*

A general function from the ambient space becomes an appropriate twisted function belonging to the dense subspace after multiplying it with a function $\psi_\epsilon(K)$ which is zero on the above described set $\mathbb{R}S_{\mathbf{B}\epsilon}$ and equal to 1 on the complement of this set. This function, $\psi_\epsilon(K)$, can be chosen to be of class C^∞ such that it is constant in radial directions satisfying $0 \leq \psi_\epsilon(K) \leq 1$, furthermore, it vanishes on $\mathbb{R}S_{\mathbf{B}\epsilon/2}$ and is equal to 1 outside of $\mathbb{R}S_{\mathbf{B}\epsilon}$. Then the $L^2_{\mathbf{Z}}$ -approximation is defined by the limiting $\epsilon \rightarrow 0$.

The twisted \mathbf{Z} -Fourier transform continuously extends to the straight \mathbf{Z} -Fourier transform defined on the ambient space. On the ambient space this Fourier transform is an automorphism. The \mathbf{Z} -Fourier transform is a bijection between the everywhere dense subspaces $\mathbf{P}\Phi_{\mathbf{B}}^n$ and $\Phi_{\mathbf{B}}^n$. Particularly the relation $\overline{\mathbf{P}\Phi_{\mathbf{B}}}^n = \overline{\Phi_{\mathbf{B}}}^n$ holds.

The same statements hold true also for Q -pole function spaces. Total spaces $\Phi_{\mathbf{B}}^{(n)}$ resp. $\mathbf{P}\Phi_{\mathbf{B}}^{(n)}$ are spanned by the corresponding Q -pole functions whose poles, Q , are in the real span of vectors belonging to \mathbf{B} .

In order to show the one to one property, suppose that the Z-Fourier transform of a twisted function vanishes. That is:

$$(69) \quad \int_{\mathbb{R}^l} e^{i\langle Z, K \rangle} \sum_{\{(p_i, q_i)\}} \phi_{(p_i, q_i)}(|X|, K) \prod_{i=1}^{k/2} z_{K_u i}^{p_i}(X) \bar{z}_{K_u i}^{q_i}(X) dK = \\ = \int_{\mathbb{R}^l} e^{i\langle Z, K \rangle} \sum_{\{(p_i, q_i)\}} \Phi_{(p_i, q_i)}(X, K) dK = 0.$$

Then, for any fixed X , the Z-Fourier transform of $\sum_{\{(p_i, q_i)\}} \Phi_{(p_i, q_i)}(X, K)$ vanishes. Therefore, this function must be zero for all X and for almost all K . For vectors K_u not lying in the singularity set $S_{\mathbf{B}}$, the \mathbf{B} is a complex basis and complex polynomials $\prod_{i=1}^{k/2} z_{K_u i}^{p_i}(X) \bar{z}_{K_u i}^{q_i}(X)$ of the X -variable are linearly independent for distinct set $\{(p_i, q_i)\}$ of exponents. Thus all component functions $\phi_{(p_i, q_i)}(|X|, K)$ of the K variable must vanish almost everywhere. This proves that the twisted Z-Fourier transform is a one to one map of the everywhere dense twisted subspace onto the everywhere dense twisted range space. The same proof, applied to straight functions, yields the statement on the straight space where the Z-Fourier transform is obviously an onto map by the well known theorem of Fourier transforms.

This section is concluded by introducing several invariant functions by which physical objects such as charge and volume will be precisely defined. Suppose that system \mathbf{B} consists of orthonormal vectors and let \mathbb{B}^\perp be the orthogonal complement of the real span, $\mathbb{B} = \text{Span}_{\mathbb{R}}(\mathbf{B})$, of vectors belonging to \mathbf{B} . Basis \mathbf{B} defines an orientation on \mathbb{B} . Let \mathbb{B}_+^\perp resp. \mathbb{B}_-^\perp be the two possible orientations which can be chosen on \mathbb{B}^\perp such that, together with the orientation of \mathbb{B} , they define positive resp. negative orientation on the X -space, \mathbb{R}^k . Charging a particle system represented by \mathbf{B} means choosing one of these two orientations. Once the system is charged, choose an orthonormal basis also in \mathbb{B}^\perp , complying with the chosen orientation. Together with \mathbf{B} , this system defines an orthonormal basis, \mathbf{Q} , on the X -space by which the matrix field $A_{ij}(Z_u)$ defined above can be introduced. The invariants of these matrix field are independent from the above basis chosen on \mathbb{B}^\perp . Invariant functions $ch(Z_u) = \text{Tr}(A_{ij}(Z_u)) - (k/2)$ and $v(Z_u) = \det(A_{ij}(Z_u))$ are called *charger* and *volumer* respectively. Their integral regarding a proper probability density defines the charge resp. mass of the particle being on a given proper state (these concepts are precisely described later in this paper).

6.3 Hankel transform.

This twisted Fourier transform is investigated by means of the Hankel transform. The statement regarding this transform asserts:

Theorem 6.2. *The Fourier transform considered on \mathbb{R}^l transforms a product, $f(r)F^{(\nu)}(\theta)$, of radial functions and spherical harmonics to the product, $H_\nu^{(l)}(f)(r)F^{(\nu)}(\theta)$, of the same form, i. e., for any fixed degree ν of the spherical harmonics, it induces maps, $H_\nu^{(l)}(f)(r)$, on the radial functions, which, so called Hankel transform, is uniquely determined for any fixed indices l and ν .*

This is actually a weak form of the original Hankel theorem which can be directly settled by the following mean value theorem of the spherical harmonics, $F^{(\nu)}(\theta)$, defined on the unit sphere, S , about the origin of \mathbb{R}^l . These functions are eigenfunctions of the Laplacian Δ_S with eigenvalue λ_ν , moreover, there exists a uniquely determined radial eigenfunction $\varphi_{\lambda_\nu}(\rho)$, where $0 \leq \rho \leq \pi$ and $\varphi_{\lambda_\nu}(0) = 1$, on S which has the same eigenvalue λ_ν such that, on a hypersphere $\sigma_\rho(\theta) \subset S$ of radius ρ and center θ on the ambient sphere S , the identity $\oint_\sigma F^{(\nu)} d\sigma_{no} = F^{(\nu)}(\theta) \varphi_{\lambda_\nu}(\rho)$ holds, where $d\sigma_{no}$ is the normalized measure measuring σ by 1.

This mean value theorem can be used for computing the Fourier transform $\int_{\mathbb{R}^l} e^{i\langle Z, K \rangle} f(|K|) F^{(\nu)}(\theta_K) dK$ at a point $Z = (|Z|, \theta_Z)$. This integral is computed by Fubini's theorem such that one considers the line $l_Z(t)$, spanned by θ_Z ; parameterized with arc-length t ; and satisfying $l_Z(Z) > 0$, and one computes the integrals first in hyperplanes intersecting l_Z at t perpendicularly and then on l_Z by dt . On the hyperplanes, write up the integral in polar coordinates defined around the intersection point with l_Z , where the radial Euclidean distance from this origin is denoted by τ . Consider polar coordinates also on S around θ_Z , where the radial spherical distance from this origin is denoted by ρ . Then $\tau = |t| |\tan \rho|$ holds, furthermore, a straightforward computation yields:

$$(70) \quad \int_{\mathbb{R}^l} e^{i\langle Z, K \rangle} f(|K|) F^{(\nu)}(\theta_K) dK = H_\nu^{(l)}(f)(r) F^{(\nu)}(\theta_Z), \quad \text{where}$$

$$H_\nu^{(l)}(f)(r) = \Omega_{l-2} \int_{-\infty}^{\infty} e^{irt} |t|^{l-1} \int_0^\pi f(|t \tan \rho|) \varphi_{\lambda_\nu}(\rho) \frac{\sin^{l-2} \rho}{|\cos \rho|^l} d\rho dt,$$

where Ω_{l-2} denotes the volume of an $(l-2)$ -dimensional Euclidean unit sphere. These formulas prove the above statement completely.

6.4 Projecting to spherical harmonics.

Among the other mathematical tools by which twisted Z-Fourier transforms are investigated are the projections $\Pi_{K_u}^{(r,s)}(\varphi(K_u) \prod_{i=1}^{\kappa} z_{K_u i}^{p_i}(X) \bar{z}_{K_u i}^{q_i}(X))$, corresponding s^{th} -order polynomials to r^{th} -order polynomials of the K_u -variable. Although the functions they are applied to may depend also on the X-variable, they refer strictly to the K_u -variable, meaning, that they are performed, over each X-vector, in the Z-space. These characteristics are exhibited also by the fact that these projections appear as certain polynomials of the Laplacian Δ_{K_u} defined on the unit K-sphere. It is much more convenient to describe them in terms of homogeneous functions, which are projected by them to harmonic homogeneous polynomials. In this version these polynomials depend on K and the projections can be described in terms of the Laplacian Δ_K defined on the ambient space. By restrictions onto the unit K-spheres, one can easily find then the desired formulas in terms of $\Pi_{K_u}^{(r,s)}$.

For an n^{th} order homogeneous polynomial, $P_n(K)$, projection $\Pi_K^{(n)} := \Pi_K^{(n,n)}$ onto the space of n^{th} order harmonic polynomials can be computed by the formula

$$(71) \quad \Pi_K^{(n)}(P_n(K)) = \sum_s C_s^{(n)} \langle K, K \rangle^s \Delta_K^s (P_n(K)),$$

where $C_0^{(n)} = 1$ and the other coefficients can be determined by the recursive formula $2s(2(s+n)-1)C_s^{(n)} + C_{s-1}^{(n)} = 0$. In fact, exactly for these coefficients is the function defined by an arbitrary $P_n(K)$ on the right side a homogeneous harmonic polynomial. These formulas can be easily established for polynomials $P_n(K) = \langle W, K \rangle^n$, defined by a fixed Z-vector W . Since they span the space of n^{th} order homogeneous polynomials, the statement follows also for general complex valued polynomials. This projection is a surjective map of the n^{th} -order homogeneous polynomial space $\mathcal{P}^{(n)}(K)$ onto the space, $\mathcal{H}^{(n)}(K)$, of the n^{th} -order homogeneous harmonic polynomials whose kernel is formed by polynomials of the form $\langle K, K \rangle P_{n-2}(K)$. Subspace $\mathcal{H}^{(n)} \subset \mathcal{P}^{(n)}$ is a complement to this kernel.

The complete decomposition of $P_n(K)$ appears in the form $P_n(K) = \sum_i \langle K, K \rangle^i H P_{n-2i}(K)$, where i starts with 0 and running through the integer part, $[n/2]$, of $n/2$, furthermore, $H P_{n-2i}(K)$ are harmonic polynomials of order $(n-2i)$. It can be established by successive application of the above computations. In the second step in this process, one considers the functions $P_n(K) - \Pi_K^{(n)}(P_n(K))$, which appear in the form $\langle K, K \rangle P_{n-2}(K)$, and obtains $H P_{n-2}(K)$ by the projection $\Pi_K^{(n-2)}(P_{n-2}(K))$. Then, also

this second term is removed from $P_n(K)$, in order to get ready for the third step, where the same computations are repeated. This process can be completed in at most $[n/2]$ steps. It is clear that projections $\Pi_K^{(n,n-2i)}$ resulting functions $HP_{n-2i}(K)$ from $P_n(K)$ are of the form $\Pi_K^{(n,n-2i)} = D_{(n,n-2i)} \Pi_K^{(n-2i,n-2i)} \Delta_K^i$, where indices $(n, n-2i)$ indicates that the projection maps n^{th} -order polynomials to $(n-2i)^{th}$ -order harmonic polynomials (in this respect, projection $\Pi_K^{(n)}$ is the same as $\Pi_K^{(n,n)}$). The technical calculation of constants $D_{(n,n-2i)}$ is omitted. The corresponding projections $\Pi_{K_u}^{(r,s)}$ defined on the unit spheres can immediately be established by these projections defined for homogeneous functions. They appear as polynomials of the Laplacian Δ_{K_u} defined on the unit K_u -sphere.

Since these projections depend just on the degrees r and s , they apply also to twisted functions which depend on the X -variable as well. In order to make the Hankel transform applicable, they are used for decomposing functions in the form

$$(72) \quad \sum_{(r;s)} f_{(r;s)}(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^{(r,s)}(\varphi(K_u)) \prod_{i=1}^{\kappa} z_{K_u i}^{p_i}(X) \bar{z}_{K_u i}^{q_i}(X) \\ = f_{\alpha}(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^{\alpha}(F^{(p_i, q_i)}(X, K_u)),$$

where the right side is just a short way to describe the sum appearing on the left side in terms of compound indices $\alpha = (r, s)$ and functions $F^{(p_i, q_i)}$.

To be more precise, functions $\phi_n(\mathbf{x}, K) P^{(n)}(X, K_u)$ appearing in the pre-spaces must be brought to appropriate forms before these projections can directly be applied to them. First of all, function $\phi_n(\mathbf{x}, K)$ should be considered in the form $\phi_n(\mathbf{x}, K) = \sum_v \phi_{n,v}(\mathbf{x}, \mathbf{k}) \varphi^{(v)}(K)$, where $\varphi^{(v)}(K)$ is an v^{th} -order homogeneous harmonic polynomial. Then, after implementing all term-by-term multiplications in the products of $\Theta_{B_i} = \langle B_i, X \rangle + \langle \mathbf{i} J_{K_u}(B_i), X \rangle$ and $\bar{\Theta}_{B_i} = \langle B_i, X \rangle - \langle \mathbf{i} J_{K_u}(B_i), X \rangle$, the above polynomials have to be taken to the form $P^{(n)}(X, K_u) = \sum_{a=0}^n P^{(n,a)}(X, K_u)$, where polynomial $P^{(n,a)}$ involves exactly a number of linear polynomials of the form $\langle J_{K_u}(Q_i), X \rangle$. The above projections defined in terms of r directly act on functions $\varphi^{(v)}(K) P^{(n,a)}(X, K)$ satisfying $r = v + a$, which are obtained by term by term multiplications of the sums given above for $\phi_n(\mathbf{x}, K)$ and $P^{(n)}(X, K_u)$.

This complicated process can be considerably simplified by considering only one-pole functions defined for single Q 's which are in the real span of the vector system \mathbf{B} . In fact, all the 1-pole total spaces $\Phi_Q^{(n)} = \sum_{p,q} \Phi_{Qpq}$,

where $n = p + q$, span also the total space $\sum_{(p_i, q_i)} \Phi_{\mathbf{B}}^{(n)}$, thus there is really enough to establish the theorems just for these simpler one-pole functions. In this case, function $P^{(n,a)}(X, K_u)$ is nothing but a constant-times of function

$$(73) \quad R_Q^{(n,a)}(X, K_u) = \langle Q, X \rangle^{n-a} \langle J_{K_u}(Q), X \rangle^a = \langle Q, X \rangle^{n-a} \langle [Q, X], K_u \rangle^a.$$

Note that, depending on p and q , the component of a particular $P^{(n)}(X, K_u)$ corresponding to a given $0 \leq a \leq n$ may vanish. However, there exist such pairs (p, q) for which this a -component is non-zero. Space $\mathbf{P}^{(n,a)}(X, K)$ is defined by the span of functions $R_Q^{(n,a)}(X, K)$. Note that K_u has been changed to K in the above formula. These functions are n^{th} - resp. a^{th} -order homogeneous polynomials regarding the X- resp. K-variables. Keep in mind that these functions can be derived from functions $\Theta_Q^p \bar{\Theta}_Q^q$, where $p + q = n$, by linear combinations, therefore, they belong to the above twisted function spaces. More precisely, for a given n , there exist an invertible matrix $M_{pq}^{(n,a)}$ such that $R_Q^{(n,a)}(X, K_u) = \sum_{p,q} M_{pq}^{(n,a)} \Theta_Q^p \bar{\Theta}_Q^q$ hold, where $p + q = n$.

Projections $\Pi_X^{(n)} = \sum_s C_s^{(n)} \langle X, X \rangle^s \Delta_X^s$ acting on functions $P(X, K_u) = \Theta_Q^p(X, K_u) \bar{\Theta}_Q^q(X, K_u)$ resp. $P(X, K_u) = \prod_{i=1}^{\kappa} z_{K_{ui}}^{p_i}(X) \bar{z}_{K_{ui}}^{q_i}(X)$, where $p + q = n$ resp. $\sum_i p_i + q_i = n$, regarding the X-variable are also involved to these investigations. Since they are n^{th} -order homogeneous functions regarding X , projection $\Pi_X^{(n)}$ applies to them immediately. Then, for any fixed K_u , function $\Pi_X^{(n)}(P(X, K_u))$ is an n^{th} -order homogeneous harmonic polynomial regarding the X-variable. The twisted Z-Fourier transforms, $\mathcal{H}\mathcal{F}_{\mathbf{B}(p_i q_i)}(\phi)(X, Z)$, involving these projections are defined by

$$(74) \quad \int_{\mathbb{R}^l} e^{i\langle Z, K \rangle} \phi(|X|, K) \Pi_X^{(n)} \left(\prod_{i=1}^{k/2} z_{K_{ui}}^{p_i}(X) \bar{z}_{K_{ui}}^{q_i}(X) \right) dK.$$

The corresponding L_Z^2 function spaces spanned by the transformed functions and the pre-space are denoted by $\Xi_{\mathbf{B}(p_i q_i)}^{(n)}$ and $\mathbf{P}\Xi_{\mathbf{B}(p_i q_i)}^{(n)}$ respectively. These function spaces are well defined also for one-pole functions and also for the third type of Z-Fourier transforms defined for Dirac type generalized functions concentrated on Z-sphere bundles.

Although the following results are not used in the rest part of this paper, because of their importance, we describe some mathematical processes by which these projections can explicitly be computed. Further on, the formulas concern a fixed n even if it is not indicated there. Decomposition into K-harmonic polynomials will be implemented for the K-homogeneous

functions $\varphi^{(v)}(K)R_Q^{(a)}(X, K)$ of order $r = v + a$. According to two different representations of the first function, these projections will be described in two different ways. The first description is more or less technical, yet, very useful in proving the independence theorems stated below. In the second description, the projected functions are directly constructed. In both cases we consider Q -pole functions defined by a unit vector $Q \in \text{Span}_{\mathbb{R}} \mathbf{B} = \mathbb{B}$. However, the multipole cases referring to vector systems \mathbf{B} are also discussed in the theorem established below.

According to the formula (74), a pure harmonic one-pole function with pole ζ in the K -space is of the form $\varphi_{\zeta}^{(v)}(K) = \sum_s D_{2s} \mathbf{k}^{2s} \langle \zeta, K \rangle^{v-2s}$, where $D_0 = |\zeta|^v$ and the other coefficients depending on $|\zeta|^{v-2s}$ and constants C_s are uniquely determined by the harmonicity assumption. It is well known that these functions span the space of homogeneous harmonic polynomials. Functions $\partial_K^c \gamma_{\zeta}^{(v)}$ obtained by directional derivatives regarding a fixed vector \tilde{K} are also homogeneous harmonic ζ -pole functions of order $(v - c)$.

The action of operator Δ_K^b on $\varphi_{\zeta}^{(v)}(K)R_Q^{(a)}(X, K)$ results function:

$$\sum_{c=0}^{\lfloor \frac{a}{2} \rfloor} D_c \langle Q, X \rangle^{n-a} (\partial_{\zeta}^{b-c} \varphi_{\zeta}^{(v)})(K) \langle J_{\zeta}(Q), X \rangle^{b-c} \langle J_K(Q), X \rangle^{a-b-c} |[Q, X]|^{2c}.$$

The terms of this sum are obtained such that Δ_K^c acts on $R_Q^{(n,a)}(X, K)$, resulting the very last term, while the others are due to the action of Δ^{b-c} on the product according to the formula $\partial_{\zeta}^{b-c} \gamma_{\zeta}^{(v)} \partial_{\zeta}^{b-c} R_Q^{(a)}(X, K)$. Note that, because of the harmonicity, the action of Δ_K on $\varphi_{\zeta}^{(v)}$ is trivial.

When the complete projection $\Pi_K^{(s)}$ is computed, then the b 's involved to the formula are denoted by b_j . For given c , factor out $|[Q, X]|^{2c}$ from the corresponding terms. Thus the final projection formula appears in the form $\sum_c |K|^{4c} P_c^{(s)}(X, K) |[Q, X]|^{2c}$, where term $P_c^{(s)}$ is equal to

$$(75) \quad \sum_j D_{cj} \langle Q, X \rangle^{n-a} (\partial_{\zeta}^{b_j-c} \varphi_{\zeta}^{(v)})(K) \langle J_{\zeta}(Q), X \rangle^{b_j-c} \langle J_K(Q), X \rangle^{a-b_j-c}.$$

This is a rather formal description of the projected functions. A more concrete construction is as follows. The linear map $\mathcal{X} \rightarrow \mathcal{Z}$ defined by $X \rightarrow [Q, X]$ is surjective whose kernel is a $(k - l)$ -dimensional subspace.

Next projections Π_K will be investigated in the Z-space over such an X-vector, \tilde{X} , which is not in this kernel and the unit vector $\zeta_Q(\tilde{X})$ defined by $[Q, \tilde{X}] = |[Q, \tilde{X}]|\zeta_Q(\tilde{X})$ is not in the singularity set $S_{\mathbf{B}}$. If X_Q denotes the orthogonal projection of X onto the l -dimensional subspace spanned by Q and vectors $J_{K_u}(Q)$ considered for all unit vectors K_u , then $|[Q, X]|^2 = |X_Q|^2 - \langle Q, X \rangle^2$.

The direct representation of $\varphi^{(v)}$ is the product of two harmonic one-pole functions having perpendicular poles. One of the poles is $\tilde{\zeta} = \zeta_Q(\tilde{X})$ while the other is an arbitrary perpendicular unit vector $\tilde{\zeta}^\perp \notin S_{\mathbf{B}}$. Then, for all $0 \leq c \leq v$, consider the product $\varphi^{(v)} = \varphi_{\tilde{\zeta}^\perp}^{(c)} \varphi_{\tilde{\zeta}}^{(v-c)}$ of harmonic homogeneous one-pole functions. Because of the perpendicular poles, these products are also harmonic functions of the K-variable, furthermore, considering them for all $0 \leq c \leq v$ and $\tilde{\zeta}^\perp$, they span the whole space of v^{th} -order homogeneous harmonic polynomials of the K-variable, for each point X . However, $\tilde{\zeta}$ is not pointing into the direction of $[Q, X]$ in general. When such a pure K-function is multiplied with $R_Q^{(n,a)}(X, K)$, over \tilde{X} , for the required projections, only the decomposition of $\varphi_{\tilde{\zeta}}^{(v-c)} \langle \tilde{\zeta}, K \rangle^a$ should be determined. A simple calculation shows:

$$(76) \quad \varphi_{\tilde{\zeta}}^{(v-c)} \langle \tilde{\zeta}, K \rangle^a = \sum_{i=0}^a D_i \langle K, K \rangle^i \varphi_{\tilde{\zeta}}^{(v-c+a-2i)}.$$

This function multiplied with $\langle Q, X \rangle^{n-a} |[Q, \tilde{X}]|^a \varphi_{\tilde{\zeta}^\perp}^{(c)}$ provides the desired decomposition and projections. When this projected function is considered over an arbitrary X , neither $\varphi_{\tilde{\zeta}^\perp}^{(c)}$ nor $\varphi_{\tilde{\zeta}}^{(v-c+a-2i)}$ appear as pure K-functions. One can state only that, for all i , this product is an a^{th} -order homogeneous polynomial, also regarding the X-variable. This complication is due to that $\zeta_Q(\tilde{X})$ and $\zeta_Q(X)$ are not parallel, thus the projection operator involves both functions to the computations. According to these arguments we have:

Theorem 6.3. (A) For given non-negative integers r , a , and $(v-a) \leq s \leq (v+a)$; where s has the same parity as $(v+a)$ or $(v-a)$; functions $\varphi^{(v)}(K) R_Q^{(n,a)}(X, K)$ project to $(v+a)^{th}$ -order K-homogeneous functions which, over \tilde{X} , appear in the form

$$(77) \quad D_{(v+a-s)/2} \langle K, K \rangle^{(v+a-s)/2} \langle Q, X \rangle^{n-a} |[Q, \tilde{X}]|^a \varphi_{\tilde{\zeta}^\perp}^{(c)} \varphi_{\tilde{\zeta}}^{(s-c)},$$

where $0 \leq c \leq v$, which, by omitting term $\langle K, K \rangle^{(v+a-s)/2}$, are s^{th} -order homogeneous K-harmonic polynomials. For given v and a , this projection is

trivial (that is, it maps to zero) for all those values s which do not satisfy the above conditions. Above, exactly the non-trivial terms are determined.

(B) Functions $\langle K, K \rangle^{(v+a-s)/2}$ and $\langle Q, X \rangle^{n-a}$ of degrees $(v+a-s)$ resp. $(n-a)$ are independent from c , furthermore, function $\varphi_{\tilde{\zeta}^\perp}^{(c)} \varphi_{\tilde{\zeta}}^{(s-c)}$ is an a^{th} -order homogeneous polynomial of the X -variable. It follows that, for a given s , the subspaces, $\mathbf{PHo}_Q^{(v,a)}$, spanned by projected functions considered for distinct pairs (v, a) are independent, more precisely, $\sum_{(v,a)} \mathbf{PHo}_Q^{(v,a)} = \mathbf{PHo}_Q^{(s)}$ is a finite direct sum decomposition of the corresponding space of complex valued homogeneous functions which are n^{th} -order regarding the X -variable and s^{th} -order harmonic functions regarding the Z -variable. This decomposition is further graded by the subspaces, $\mathbf{PHo}_Q^{(v,a,c)}$ defined for distinct c 's introduced above.

(C) These statements remain true for the total spaces $\mathbf{PHo}_B^{(v,a)}$ and $\mathbf{PHo}_B^{(s)}$ obtained by summing up all the corresponding previous spaces defined for Q 's which are in the real span of independent vector-system \mathbf{B} . Since the projections Π_X and Π_K commute, they remain true also for spaces $\mathbf{PXo}_B^{(v,a)}$ and $\mathbf{PXo}_B^{(s)}$ (as well as for versions defined for fixed Q 's) obtained by applying Π_X to the corresponding spaces \mathbf{PHo} .

Proof. The independence-statement in (B) follows also from (75), because functions $||Q, X||^{2c}$ regarding distinct powers $2c$ are independent and, for a fixed \tilde{K}_u , also functions $P_c^{(s)}(X, \tilde{K})$ of the X -variable expressed by means of complex structure $J_{\tilde{K}_u}$ as a complex valued function has distinct real and imaginary degrees with respect to distinct a 's. That is, these functions defined for a fixed a can not be a linear combination of the others defined for other a 's.

Statement (C) can be established by an appropriate generalization of (75). Instead of Q -pole functions, now functions belonging to $\mathbf{PHo}_B^{(v,a_i)}$ should be projected, where $a = \sum a_i$ and degree a_i regards B_i . In this situation, one gets functions $\langle [B_i, X], [B_j, X] \rangle^{v_{ij}}$ multiplied with the corresponding functions $P_{c_{ij}}^{(s)}(X, V)$, which, for a fixed \tilde{K}_u and system c_{ij} of exponents have distinct real and imaginary degrees regarding $J_{\tilde{K}_u}$ with respect to distinct $\sum_i a_i = a \neq a' = \sum_i a'_i$. \square

6.5 Twisted Hankel decomposition.

In order to construct the complete pre-spaces $\mathbf{P}\Phi_Q^{(n)}$, $\mathbf{P}\Phi_B^{(n)}$, $\overline{\mathbf{P}\Phi_Q^{(n)}}$, and $\overline{\mathbf{P}\Phi_B^{(n)}}$, by means of functions in $\mathbf{PHo}_Q^{(n,v,a)}$ resp. $\mathbf{PHo}_B^{(n,v,a)}$, they must be multiplied with functions of the form $\phi(|X|, |K|)$ which multiplied with $|K|^{v+a}$ incorporated to the \mathbf{PHo} -functions provide K-radial L_K^2 -functions for any fixed $|X| = \mathbf{x}$. These functions summed up regarding (v, a) span the corresponding n^{th} -order twisted spaces whose L_K^2 -closures provide the complete space which can also be introduced by straightly defined functions. Since they are everywhere dense in the straightly defined function spaces, the twisted spaces must be complete regarding the \mathbf{PHo} -spaces. Thus the completion of twisted spaces is ultimately implemented on the space of K-radial functions.

Projections $\Pi_K^{(s)}$ should be defined for functions belonging to $\mathbf{P}\Phi_Q^{(n,v,a)}$ resp. $\mathbf{P}\Phi_B^{(n,v,a)}$, first. Note that this operation has no effect on radial functions. It's action is restricted to the spherical harmonics defined by restricting the above homogeneous K-harmonic polynomials to the unit K-sphere over each point X . By this interpretation, these projection operators can be expressed as polynomials of the Laplacian Δ_{K_u} defined on this unit sphere. The function spaces obtained by projecting the whole corresponding ambient spaces are denoted by $\mathbf{P}\Phi_Q^{(n,s)}$ resp. $\mathbf{P}\Phi_B^{(n,s)}$. There is described in the previous theorem that which functions labeled by (v, a) provide non-trivial s -components in these operations. They provide direct sum decompositions of the ambient spaces which is called also *pre-Hankel decomposition*. When also $\Pi_X^{(n)}$ (which operator commutes with $\Pi_K^{(s)}$, for all s) is acting, the obtained pre-Hankel spaces are denoted by $\mathbf{P}\Xi_Q^{(n,s)}$ resp. $\mathbf{P}\Xi_B^{(n,s)}$, where n still indicates the degree of the involved homogeneous polynomials regarding the X -variable.

The Z-Fourier transforms $\int e^{i\langle Z, K \rangle} \mathbf{P}\Phi_Q^{(n,s)} dK$ resp. $\int e^{i\langle Z, K \rangle} \mathbf{P}\Phi_B^{(n,s)} dK$ define the s^{th} -order twisted Hankel spaces $\mathbf{H}\Phi_Q^{(n,s)}$ resp. $\mathbf{H}\Phi_B^{(n,s)}$, which are projected to $\mathbf{H}\Xi_Q^{(n,s)}$ resp. $\mathbf{H}\Xi_B^{(n,s)}$ by $\Pi_X^{(n)}$. The direct sums of these non-complete subspaces define the corresponding *total twisted Hankel spaces*. They are different from the pre-spaces, but also everywhere dense subspaces of the corresponding straightly defined function spaces. Thus the closure of these twisted spaces provides again the whole straightly defined spaces.

6.6 Twisted Dirichlet and Z-Neumann functions.

All above constructions are implemented by using the whole center \mathbb{R}^l . In this section twisted functions satisfying the Dirichlet or Z-Neumann condition on the boundary, ∂M , of ball \times ball- type domains, M , are explicitly constructed by the method described in the review of this section. That is, they are represented by twisted Z-Fourier transforms of Dirac type generalized functions concentrated on ∂M . Due to this representation, the eigenfunctions of the exterior operator \mathbb{E} satisfying given boundary conditions can explicitly be computed.

In the first step of this process consider a sphere, S_R , of radius R around the origin of the Euclidean space \mathbf{R}^l . As it is well known, the Dirichlet or Neumann eigenfunctions of the Euclidean Laplacian $-\Delta_{\mathbb{R}^l}$ on the ball B_R bounded by S_R appear as products of s^{th} -order spherical harmonics $\varphi^{(s)}(K_u)$ with radial functions $y_i^{(s)}(\mathbf{z})$. For $s = 0$, these eigenfunctions are radial taking 1 at the origin and having multiplicity 1. For $s > 0$, the radial functions take 0 at the origin and the multiplicity, for fixed s and i , is equal to the dimension of the space of s^{th} -order spherical harmonics $\varphi^{(s)}(K_u)$. Corresponding to the Dirichlet or Neumann conditions, these eigenvalues are denoted by $\lambda_{Di}^{(s)}$ and $\lambda_{Ni}^{(s)}$ respectively. For any fixed s and condition D or N , these infinite sequences satisfy $\lambda_i^{(s)} \uparrow \infty$ and, except for $0 = \lambda_{N1}^{(0)}$, also the relation $0 < \lambda_i^{(s)}$ holds.

Eigenfunctions corresponding to Dirichlet or Neumann eigenvalues $\lambda_i^{(s)} = \lambda$ can be represented by the integral formula

$$(78) \quad y_i^{(s)}(\mathbf{z})\varphi^{(s)}(Z_u) = \oint_{S_{\sqrt{\lambda}}} e^{i\langle Z, V \rangle} \varphi^{(s)}(K/\mathbf{k}) dK_{no},$$

where dK_{no} is the normalized integral density on the sphere of radius $\sqrt{\lambda}$. Apply $-\Delta_{\mathbb{R}^l}$ on the right side to see that this function is an eigenfunction of this operator with eigenvalue $\lambda_i^{(s)} := \lambda$, which, because of the Hankel transform, must appear as the function being on the left side. This formula strongly relates to the third version of the twisted Z-Fourier transforms which is introduced also in the review of this section. However, it can be directly applied there only after decomposing the functions behind the integral sign by the projections $\Pi_{K_u}^{(r,s)}$.

Before this application, functions $y_i^{(s)}(\mathbf{z}) := y(t)$, belonging to an eigenvalue λ , are more explicitly determined as follows. By formulas

$$(79) \quad \Delta_Z = \partial_t \partial_t + \frac{l-1}{t} \partial_t + \frac{\Delta_S}{t^2}, \quad \Delta_S \varphi^{(s)} = -s(s+l-2) \varphi^{(s)},$$

it satisfies the differential equation

$$(80) \quad y'' + \frac{l-1}{t} y' + \left\{ \lambda - \frac{s(s+l-2)}{t^2} \right\} y = 0,$$

which, after the substitutions $\tau = \sqrt{\lambda} t$ and $y(t) = z(\tau)$, becomes

$$(81) \quad z'' + \frac{l-1}{\tau} z' + \left\{ 1 - \frac{s(s+l-2)}{\tau^2} \right\} z = 0.$$

That is, function $J(\tau) = \tau^{l/2-1} z(\tau)$ satisfies the ordinary differential equation

$$(82) \quad J'' + \frac{1}{\tau} J' + \left\{ 1 - \frac{(2s+l-2)^2}{4\tau^2} \right\} J = 0,$$

therefore, it is a bounded Bessel function of order $(s+l/2-1)$. Thus, except for complex multiplicative constant, equation $J = J_{s+l/2-1}$ must hold.

In order to find the functions satisfying the boundary conditions, consider spheres $S_{\lambda(\mathbf{x}^2)}$ of radius $\lambda(\mathbf{x}^2)$ around the origin of the Z-space. For appropriate functions $\phi(\mathbf{x}^2)$ and $\varphi(K_u)$, the twisted Z-Fourier transform on the sphere bundle S_λ is defined by:

$$(83) \quad \mathcal{F}_{Qpq\lambda}(\phi\varphi)(X, Z) = \oint_{S_{\sqrt{\lambda}}} e^{i\langle Z, K \rangle} \phi(\mathbf{x}^2) \varphi(K_u) (\Theta_Q^p \overline{\Theta}_Q^q)(X, K_u) dK_{no},$$

where $dK_{no} = dK/Vol(\sigma_\lambda)$ is the normalized measure on the sphere and function $\varphi(K/\mathbf{k}) \Theta_Q^p \overline{\Theta}_Q^q(X, K/\mathbf{k})$ is defined on S_λ . Since no Hankel projections are involved, these functions do not satisfy the required boundary conditions, yet. However, by the above arguments we have:

Theorem 6.4. *Consider a ball \times ball-type domain defined by the Z-balls $B_{R(\mathbf{x}^2)}$ and let $\lambda_i^{(s)}(\mathbf{x}^2) := \lambda(\mathbf{x}^2)$ be a smooth function defined by the i^{th} eigenvalue in the s^{th} -order Dirichlet or Neumann spectra of the Euclidean balls $B_{R(\mathbf{x}^2)}$. Then function (83) defined for $\lambda = \lambda_i^{(s)}(\mathbf{x}^2)$, or*

$$(84) \quad \oint_{S_{\sqrt{\lambda}}} e^{i\langle Z, K \rangle} \phi(|X|^2) \pi_K^{(s)}(\beta^{(m)} \Pi_X^{(n)}(\Theta_Q^p \overline{\Theta}_Q^q))(X, K_u) dK_{no}$$

satisfy the Dirichlet resp. Z-Neumann condition on the domain M . The restrictions of these functions onto M span $\Phi_Q^{(n,s)}(M)$ resp. $\Xi_Q^{(n,s)}(M)$, for any fixed boundary condition. If functions $\Theta_Q^p \overline{\Theta}_Q^q$ are exchanged for the polynomials $\prod_{i=1}^{\kappa} z_{K_{ui}}^{p_i}(X) \overline{z}_{K_{ui}}^{q_i}(X)$, then the above construction provides the

Dirichlet-, Z-Neumann-, resp. mixed-condition-functions spanning $\Phi_{\mathbf{B}}^{(n,s)}(M)$
resp. $\Xi_{\mathbf{B}}^{(n,s)}(M)$.

6.7 Constructing the orbital and inner force operators.

The complicated action of the compound angular momentum operator \mathbf{M}_Z on twisted Hankel functions is due to fact that its Fourier transform, $\mathbf{i}D_K\bullet$, is non-commuting with the Hankel projections, that is, the commutator on the right side of $D_K \bullet \Pi_{K_u}^\alpha = \Pi_{K_u}^\alpha D_K \bullet + [D_K \bullet, \Pi_{K_u}^\alpha]$ is non-vanishing in general. Thus there are two non-trivial terms on the right side of equation

$$\begin{aligned} (85) \quad \mathbf{M}_Z \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK &= \\ &= \mathbf{i} \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) D_K \bullet \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK = \\ &= \mathbf{i} \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) (\Pi_{K_u}^\alpha D_K \bullet + [D_K \bullet, \Pi_{K_u}^\alpha]) (F^{(p_i, q_i)}(X, K_u)) dK, \end{aligned}$$

by which the orbital:

$$\begin{aligned} (86) \quad \mathbf{L}_Z \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK &= \\ &= \mathbf{i} \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^\alpha D_K \bullet (F^{(p_i, q_i)}(X, K_u)) dK \end{aligned}$$

and the intrinsic spin operators:

$$\begin{aligned} (87) \quad \mathbf{S}_Z \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK &= \\ &= \mathbf{i} \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) [D_K \bullet, \Pi_{K_u}^\alpha] (F^{(p_i, q_i)}(X, K_u)) dK \end{aligned}$$

are defined, respectively.

The commutator appears in the following explicit form

$$(88) \quad [D_K \bullet, \Pi_{K_u}^\alpha] = S_\beta^\alpha \Pi_{K_u}^\beta \mathbf{M}_{K_u^\perp} = \mathbf{M}_{K_u^\perp} S_\beta^\alpha \Pi_{K_u}^\beta,$$

where $\mathbf{M}_{K_u^\perp} = \mathbf{M}_K - \partial_{\mathbf{k}} D_{K_u} \bullet$ is the Z-spherical angular momentum operator defined on Z-spheres. (For a fixed unit vector K_u and orthonormal basis $\{e_\alpha, K_u\}$ (where $\alpha = 1, 2, \dots, l-1$) of K-vectors, this operator is of the form $\mathbf{M}_{K_u^\perp} = \sum_\alpha \partial_\alpha D_\alpha \bullet$.) Formula (88) follows by applying relations $D_K \bullet \Delta_{K_u} = \Delta_{K_u} D_K \bullet - 2\mathbf{M}_{K_u^\perp}$ and the commutativity of Δ_{K_u} with operators $\partial_\alpha, \partial_{K_u}, D_\alpha \bullet$, and $D_{K_u} \bullet$ in order to evaluate

$$D_K \bullet \Pi_{K_u}^{(r, r-2i)} = D_K \bullet \tilde{D}_{(r, r-2i)} \Pi_K^{(r-2i, r-2i)} \Delta_{K_u}^i,$$

where, according to the denotations introduced above, $r = v + a$ holds. This computation results the equation $[D_K \bullet, \Pi_{K_u}^{(r, r-2i)}] = P_{i+1}(\Delta_{K_u}) \mathbf{M}_{K_u^\perp}$, where the lowest exponent of Δ_{K_u} in the polynomial $P_{i+1}(\Delta_{K_u})$ is $i+1$. This term defines a uniquely determined constant times of projection $\Pi_{K_u}^{(r, r-2(i+1))}$ such that $P_{i+1}(\Delta_{K_u}) = A_{i+1} \Pi_{K_u}^{(r, r-2(i+1))} + P_{i+2}(\Delta_{K_u})$ holds, where the lowest exponent of Δ_{K_u} in the polynomial $P_{i+2}(\Delta_{K_u})$ is $i+2$. In the next step, the above arguments are repeated regarding P_{i+2} to obtain P_{i+3} . This process can be concluded in finite many steps which establish the above formula completely.

Formula (88) allows to define an *inner algorithm* where these computations are iterated infinitely many times. In the second step it is repeated for $f_\beta^{(2)} := S_\beta^\alpha f_\alpha$ as follows. First note that operator $f_\alpha S_\beta^\alpha \Pi_{K_u}^\beta \mathbf{M}_{K_u^\perp} = \mathbf{M}_{K_u^\perp} \Pi_{K_u}^\beta S_\beta^\alpha f_\alpha$ can be decomposed in the following form:

$$(89) \quad f_\beta^{(2)} \Pi_{K_u}^\beta \mathbf{M}_{K_u^\perp} = -\partial_{K_u} (f_\beta^{(2)}) \Pi_{K_u}^\beta D_{K_u} \bullet + \mathbf{M}_K \Pi_{K_u}^\beta f_\beta^{(2)}.$$

The action of the first operator on functions

$$(90) \quad F^{(p,q)}(X, K_u) = \varphi_\zeta^{(r)}(K_u) \Theta_Q^p(X, K_u) \overline{\Theta}_Q^q(X, K_u)$$

$$(91) \quad \text{resp.} \quad F^{(p,q)}(X, K_u) = \varphi_\zeta^{(r)}(K_u) \prod_{i=1}^{k/2} z_{K_u i}^{p_i} \bar{z}_{K_u i}^{q_i},$$

where $\varphi_\zeta^{(r)}(K_u)$ denotes an r^{th} -order homogeneous harmonic polynomial introduced at explicit description of Hankel projections, results

$$-(p-q) \mathbf{i} \partial_{K_u} (f_\alpha S_\beta^\alpha) \Pi_{K_u}^\beta (F^{(p,q)}).$$

This term is already in finalized form which does not alter during further computations. Together with the orbiting operator, they define, in terms of the radial operator

$$(92) \quad \bigcirc_\alpha^{(1)}(f_{\beta_1}, \dots, f_{\beta_d}) = -(p-q) \mathbf{i} (\mathbf{k} f_\alpha + \partial_{K_u} (f_\beta S_\alpha^\beta)),$$

the one-turn operator

$$(93) \quad \mathbf{M}_Z^{(1)} \int e^{\mathbf{i}\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK = \\ = \int e^{\mathbf{i}\langle Z, K \rangle} \bigcirc_\alpha^{(1)}(f_{\beta_1}, \dots, f_{\beta_d}) \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK,$$

where the name indicates that it is expressed in terms of the first power of the inner spin operator S_α^β permutating the Hankel radial functions.

Be aware of the novelty of this spin-concept emerging in these formulas! It defines just permutation of the radial Hankel functions to which no actual

spinning of the particles can be corresponded. This concept certainly does not lead to a dead-end-theory like those pursued in classical quantum theory where one tried to explain the inner spin of electron by actual spinning.

This abstract merry-go-round does not stop after making one turn. It is actually the second operator, $\mathbf{M}_K \Pi_{K_u}^\beta f_\beta^{(2)}$, in (89) which generates the indicated process where the above arguments are repeated for functions $f_\beta^{(2)} := S_\beta^\alpha f_\alpha$. Index 2 indicates that these functions are obtained from the starting functions $f_\alpha^{(1)} := f_\alpha$ by the next step. The details are as follows.

In these computations the second operator is derived by the Hankel transform turning functions defined on the \tilde{K} -space to functions defined on the K -space. Operator $H^{(-s_\beta)}$ denotes the inverse of the Hankel transform $H_{s_\beta}^{(l)}$, where s_β denotes the third index in $\beta = (v, a, s)$. Then we have:

$$\begin{aligned}
 (94) \quad & \mathbf{M}_K f_\beta^{(2)} \Pi_{K_u}^\beta F^{(p,q)}(X, K_u) = \\
 & = \mathbf{M}_K \int e^{i\langle K, \tilde{K} \rangle} H^{(-s_\beta)}(f_\beta^{(2)})(\mathbf{x}, \tilde{\mathbf{k}}) \Pi_{\tilde{K}_u}^\beta F^{(p,q)}(X, \tilde{K}_u) d\tilde{K} = \\
 & = \mathbf{i} \int e^{i\langle K, \tilde{K} \rangle} \tilde{f}_\beta^{(2)}(\mathbf{x}, \tilde{\mathbf{k}}) D_{\tilde{K}} \bullet \Pi_{\tilde{K}_u}^\beta F^{(p,q)}(X, \tilde{K}_u) d\tilde{K},
 \end{aligned}$$

where $\tilde{f}_\beta^{(2)} = H^{(-s_\beta)}(f_\beta^{(2)})$. At this step commutator $[D_{\tilde{K}} \bullet, \Pi_{\tilde{K}_u}^\beta]$ can be calculated by (88), resulting functions $f_\beta^{(3)} := S_\beta^\alpha \tilde{f}_\alpha^{(2)}$ which are subjected to the operations performed in the following step 3. These steps must infinitely many times be iterated.

Regarding radial functions $\tilde{f}_\alpha^{(2)}$, which are defined on the \tilde{K} -space, the orbiting spin and the one-turn operator can be defined in the same way as they are defined on the K -space. After performing the Z-Fourier and the associated Hankel transforms on these finalized functions they become functions defined on the K -space. By adding these terms to those obtained in the first step, one defines the two-turn operator $\mathbf{M}_Z^{(2)}$ associated with $\bigcirc_\alpha^{(2)}(f_{\beta_1}, \dots, f_{\beta_d})$, called two-turn merry-go-round and roulette operators respectively. The sum of the orbiting spin operators defines $\mathbf{L}^{(2)}$, which is called one-turn orbiting spin operator involving just S_β^α into its definition. One should keep in mind that these operators involve also the operators defined in the previous step. These computations work out for an arbitrary u^{th} -step defining operators $\mathbf{M}_Z^{(u)}$, $\bigcirc_\alpha^{(u)}(f_{\beta_1}, \dots, f_{\beta_d})$, and $\mathbf{L}^{(u)}$ which are called u -turn merry-go-round-, roulette-, and $(u-1)$ -turn orbiting spin operator respectively. In the end, the action of $\mathbf{M}_Z = \mathbf{M}_Z^{(\infty)}$ can be described

in the form:

$$(95) \quad \mathbf{M}_Z \int e^{i\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK = \\ = \int e^{i\langle Z, K \rangle} \bigcirc_\alpha (f_{\beta_1}, \dots, f_{\beta_d}) \Pi_{K_u}^\alpha (F^{(p_i, q_i)}(X, K_u)) dK,$$

where operator $\bigcirc_\alpha(f_{\beta_1}, \dots, f_{\beta_d}) = \bigcirc_\alpha^{(\infty)}(f_{\beta_1}, \dots, f_{\beta_d})$, called high roulette operator, is defined by the infinite series $\lim_{u \rightarrow \infty} \bigcirc_\alpha^{(u)}$. In this sense, the complete angular momentum operator \mathbf{M}_Z can be called high merry-go-round operator. The point in this formula is that this action can be described in terms of d-tuples, $(f_{\beta_1}, \dots, f_{\beta_d})$, of radial functions which can not be reduced to a single function defined by a fixed α .

This statement holds also for the total Laplacian Δ , thus the eigenfunctions satisfying the Dirichlet or Z-Neumann conditions on the considered manifolds can also be completely described in terms of radial functions. But the corresponding equations involve all radial functions $(f_{\beta_1}, \dots, f_{\beta_d})$ defined for all indices β_i . By this reason, operator \bigcirc_α plays the role of a confider, giving rise to the effect that the eigenfunctions satisfying a given boundary condition are expressed in terms of radial functions which do not satisfy these conditions individually but exhibit them together, confined in a complicated combination. Neither can these eigenfunctions be observed as single \mathbb{E} -force potential functions. On the other hand, for a given boundary condition, the \mathbb{E} -eigenfunctions span the whole L^2 -space, therefore, Δ -eigenfunctions satisfying the same boundary condition can be expressed as infinite linear combinations of \mathbb{E} -eigenfunctions. This observation further supports the idea that the rather strong Δ -forces are built up by the much weaker \mathbb{E} -forces in a way how Hawking describes the action of the Weinberg-Salam weak force in [H], pages 79:

“The Weinberg-Salam theory known as spontaneous symmetry breaking. This means that what appear to be a number of completely different particles at low energies are in fact found to be all the same type of particle, only in different states. At high energies all these particles behave similarly. The effect is rather like the behavior of a roulette ball on a roulette wheel. At high energies (when the wheel is spun quickly) the ball behaves in essentially only one way – it rolls round and round. But as the wheel slows, the energy of the ball decreases, and eventually the ball drops into one of the thirty-seven slots in the wheel. In other words, at low energies there are thirty seven different states in which the ball can exist. If, for some reason, we could only observe the ball at low energies, we would then think that there were thirty-seven different types of ball!”

This quotation explains the origin of the name given to the roulette operators. This polarized operators are derived from the non-polarized merry-go-round operators, which name was suggested to me by Weinberg's book [W1] where the name "merry-go-round" appears in a different situation not discussed here. All these arguments clearly suggest that the Δ -eigenfunctions must correspond to the strong forces keeping the quarks together. A formula expressing these eigenfunctions as linear combinations of weak force potential functions would shed light to the magnitude of the strong force piled by the roulette operator during building up these eigenfunctions.

The explicit eigenfunction computations give rise to difficult mathematical problems which are completely open at this point of the developments. In this paper we explicitly describe only the eigenfunctions of the exterior orbiting operator \mathbb{E} defined by omitting the interior spin operator from the Δ . This operator is a scalar operator whose action can be reduced to a single radial function f . By the arguments developed in the introductions and also at several points in the body of this paper, this operator is associated with the weak force interaction. These computations are established in the next section. This section is concluded by saying more about the recovery of several concepts of the standard model within this new theory.

Suppose that function

$$(96) \quad \psi(X, Z) = \int e^{i\langle Z, K \rangle} f_\alpha(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^\alpha(F^{(p_i, q_i)}(X, K_u)) dK$$

is an eigenfunction of the complete Laplacian Δ . It is called also probability amplitude of the particle system. Also in this formula the Einstein convention indicates summation. A fixed index $\alpha = (v, a, s)$ is called slot-index and function

$$(97) \quad \mathcal{Q}_{vas}(X, Z) = \int e^{i\langle Z, K \rangle} f_{vas}(\mathbf{x}, \mathbf{k}) \Pi_{K_u}^{(vas)}(F^{(p_i, q_i)}(X, K_u)) dK$$

is the so called high slot probability amplitude. These exact mathematical objects correspond to quarks whose flavor is associated with index v and its color is associated with index a . Index s is called azimuthal index. Note that also these indices have mathematical meanings, namely, they refer to the degrees of the corresponding polynomials by which the formula of a slot-amplitude is built up. According to these definitions, a particle-system-amplitude is the sum of the high slot probability amplitudes which are considered to be the mathematical manifestations of quarks. Slot amplitudes defined in the same way by the \mathbb{E} -eigenfunctions are called retired or laid-off slot probability amplitudes.

It is very interesting to see how these constituents of a high energy particle are held together by the strong force interaction. When $D_{K\bullet}$ is acting

on functions $\langle Q, X \rangle$ resp. $\langle J_K(Q), X \rangle$ of a quark, then the first one becomes of the second type and the second one becomes of the first type. In either cases the color index, a , changes by 1 and a slot-amplitude of odd color index becomes an amplitude of even color index. This process can be interpreted as gluon exchange in the following way. Action of $D_{K\bullet}$ on $\langle Q, X \rangle$ resp. $\langle J_K(Q), X \rangle$ are interpreted as gluon absorption resp. emission. More precisely, some of the slot-particles (quarks) of odd color index emit gluon which is absorbed by some of the slot-particles (quarks) which also have odd color index. As a result they become quarks of even color index. The same process yields also for quarks having even color index, which, after gluon exchange, become quarks of odd color index. This is a rather clear explanation for the flavor-blindness and color sensitiveness of gluons.

It also explains why a high slot-particle (quark) can never retire to become a laid-off \mathbb{E} -particle. Indeed, in a high slot-state defined for fixed slot index $\alpha = (v, a, s)$ the corresponding Hankel function does not satisfy the boundary conditions, however, it can be expanded by the \mathbb{E} -eigenfunctions. At this point nothing is known about the number of \mathbb{E} -eigenfunctions by which a high slot-amplitude can be expressed. This number can very well be equal to the infinity, but it is always greater than one. Then, instead of consisting of a single term, the high slot probability density is a multi-term sum of weak densities determined by these laid off probability amplitudes.

Real positive function $\psi\bar{\psi}$ whose integral on the whole ball \times ball-type domain is 1 is called probability density. Protons are defined by functions $ch(Z_u)(\psi_c\bar{\psi}_c)(Z, X)$, where ψ_c is a constant times of ψ , whose integral on the ball \times ball-type domain is 1. If this integral is 0 or -1 , it is called neutron or antiproton respectively. (The mass can be defined by means of $|\det(A_{ij}(Z_u))|$, but we do not go into these details here.) This argument shows that in a complete eigensubspace decomposition of the L^2 function space of a two-step nilpotent Lie group representing a particle system the eigenfunctions, actually, represent all kind of particles and not just particular ones. This phenomenon can be considered as a clear manifestation of the bootstrap principle, from which super string theory grew out, also in the new theory. In super string theory the notion was (cf. [G], pages 128) that “a set of elementary particles could be treated as if composed in a self-consistent manner of combinations of those same particles. All the particles would serve as constituents, all the particles (even the fermions in a certain sense) would serve as quanta for force fields binding the constituents together, and all the particles would appear as bound states of the constituents”.

6.8 \mathbb{E} -forces in the union of 3 fundamental forces.

The eigenfunction computations of \mathbb{E} on the twisted function space $\Xi_R^{(n)}$ satisfying a given boundary condition can be reduced to the same radial differential operator what was obtained for the standard Ginsburg-Landau-Zeeman operator on Z-crystals. To see this statement, let the complete Laplacian (32) act on (84). By the commutativity relation $\mathbf{M}_Z \mathbb{f} = \mathbb{f} \mathbf{i} D_K \bullet$ and $\Delta_Z \mathbb{f} = -\mathbb{f} \mathbf{k}^2$, this action is a combination of X-radial differentiation, $\partial_{\mathbf{x}}$, and multiplications with functions depending just on \mathbf{x} , that is, it is completely reduced to X-radial functions. More precisely,

$$(98) \quad \Delta(\mathcal{H}\mathcal{F}_{QpqR}(\phi\beta)) = \oint_{S_R} e^{i\langle Z, K \rangle} \diamond_{R, \mathbf{x}^2}(\phi) \beta \Pi_X^{(n)}(\Theta_Q^p \bar{\Theta}_Q^q) dK_{no}$$

holds, where

$$(99) \quad (\diamond_{R, \mathbf{x}^2} \phi)(\mathbf{x}^2) = 4\mathbf{x}^2 \phi''(\mathbf{x}^2) + (2k + 4(p + q))\phi'(\mathbf{x}^2) + \\ - (R^2(1 + \frac{1}{4}\mathbf{x}^2) + (p - q)R)\phi(\mathbf{x}^2),$$

and ϕ', ϕ'' mean the corresponding derivatives of $\phi(\tilde{t})$ with respect to the \tilde{t} variable. The eigenfunctions of Δ can be found by seeking the eigenfunctions of the reduced operator $\diamond_{R, \mathbf{x}^2}$ among the X-radial functions.

Note that no projections $\Pi_K^{(s)}$ are applied in the above integral formula, thus these eigenfunctions do not satisfy the boundary conditions in general. These projections, however, do not commute with $D_K \bullet$, and the eigenfunctions of the complete operator Δ can not be expressed in terms of a single function $\phi(\mathbf{x}^2)$. This simple reduction applies just to the exterior operator, \mathbb{E} , defined by neglecting the anomalous intrinsic momentum operator \mathbf{S}_Z from Δ and keeping just the orbital (alias, exterior) spin operator \mathbf{L}_Z . Regarding this operator we have:

Theorem 6.5. *The exterior operator \mathbb{E} , on constant radius Z-ball bundles reduces to a radial operator appearing in terms of the Dirichlet-, Neumann-, resp. mixed-condition-eigenvalues $\lambda_i^{(s)}$ of the Z-ball $B_Z(R)$ in the form*

$$(100) \quad (\diamond_{\lambda, \tilde{t}} f)(\tilde{t}) = 4\tilde{t} f''(\tilde{t}) + (2k + 4n) f'(\tilde{t}) - (2m \sqrt{\frac{\lambda_i^{(s)}}{4}} + 4 \frac{\lambda_i^{(s)}}{4} (1 + \frac{1}{4}\tilde{t})) f(\tilde{t}).$$

By the substitution $\lambda = \sqrt{\lambda_i^{(s)}/4}$, this is exactly the radial Ginsburg-Landau-Zeeman operator (55) obtained on Z-crystal models.

Despite this formal identity with electromagnetic forces, the nuclear forces represented by \mathbb{E} manifest themselves quite differently. Like for the electromagnetic forces, one can introduce both charged and neutral particles also

regarding \mathbb{E} . A major difference between the two particle-systems is that the particles represented by \mathbb{E} are extended ones. This property can be seen, for instance, by the eigenfunctions of Δ which involve also Z -spherical harmonics. Due to these harmonics, the multiplicities of the eigenvalues regarding \mathbb{E} are higher than what is corresponded to the same eigenvalue regarding the Ginsburg-Landau-Zeeman operator on Z -crystals. An other consequence of the extension is that the \mathbb{E} -neutrinos always have positive mass, which is zero for neutrinos associated with electromagnetic forces.

Actually, the nuclear forces represented by \mathbb{E} are weaker than the electromagnetic forces. This phenomenon can be explained, by the extension, by the very same argument of classical electrodynamics asserting that the electromagnetic self-mass for a surface distribution of charge with radius a is $e^2/6\pi ac^2$, which, therefore, blows up for $a \rightarrow 0$. Note that, in the history of quantum theory, this argument provided the first warning that a point electron will have infinite electromagnetic self-mass. It is well known that this problem appeared with even much greater severity in the problem of infinities invading quantum field theory. The tool by which these severe problems had been handled is renormalization, which turned QED into a renormalizable theory. The above argument clearly suggests that also the \mathbb{E} -theory must be renormalizable.

The major difference between the \mathbb{E} - and the complete Δ -theory is that the first one is a scalar theory, which can be reduced to a radial operator acting on a single radial function, while the reduced operator obtained in the Δ -theory acts on d -tuples of radial functions. As it is described above, this action defines also a new type of nuclear inner spin of the extended particles to which new particles such as quarks and gluons can be associated and by which strong nuclear forces keeping the parts of the nucleus together can be introduced. Partial operator \mathbb{E} is the maximal scalar operator in Δ . Except for $\mathbb{E} = \mathbb{E}^{(1)}$, partial operators $\mathbb{E}^{(u)}$ (defined by replacing \mathbf{M}_Z by $\mathbf{L}_Z^{(u)}$) and $\Delta^{(u)}$ (defined by replacing \mathbf{M}_Z by $\mathbf{M}_Z^{(u)}$) can be reduced just to operators which irreducibly act on d -tuples of radial functions.

The main unifying principle among the three fundamental forces is that they are derived from the very same Hamilton operator, Δ . More precisely, the Hamilton operators of the individual elementary particles emerge on corresponding invariant subspaces of Δ such that it is restricted to the subspace corresponded to the given elementary particles. The corresponding forces are defined by those acting among these particles. The electromagnetic forces manifest themselves on function spaces consisting functions which are periodic regarding a Z -lattice Γ_Z . The systems of particles defined on these function spaces are without interior. They consist of particles such as

electrons, positrons, and electron-positron-neutrinos. The various nuclear forces appear on function spaces defined on Z-ball and Z-sphere bundles by fixed boundary conditions. The attached particles, which do have interior, and the forces acting among them are discussed above.

The function spaces corresponded to particular particles are constructed with the corresponding twisted Z-Fourier transforms. Since the eigenfunctions of the Hamilton operators can be sought in this form, this transform seems to be the only natural tool for assigning the invariant function spaces corresponding to elementary particles. It is also remarkable that the twisted Z-Fourier transforms emerge also in the natural generalization of the de Broglie waves fitting the new theory.

A particle-system defined by a fixed function space consists of all kind of particles which can be defined by the characteristic property of the given function space. For instance, all particles without interior appear on a Γ_Z -periodic function spaces. In order to avoid the annihilation of particles by antiparticles, the particles belonging to the same system are considered to be not interacting with each other. According to this argument, the particles in a system defined by an invariant subspace are gregarious which are always accompanied with other particles. For instance, an electron is always partying with an electron-neutrino. This complexity of the particle-systems is reflected by the Laplacian (Hamiltonian) which appears as the sum of Hamiltonians of particles partying in a system. This phenomenon is a relative of those described by the bootstrap principle of super string theory.

Let it be mentioned yet that the distinct function spaces belonging to distinct type of forces are not independent, thus there is the possibility to work out an interaction theory between particles of distinct types. The existence of such a viable theory is a completely open question in this field.

The \mathbb{E} -forces are very similar to the weak nuclear forces described in the Weinberg-Salam theory. Weinberg introduces these forces on pages 116-120 of his popular book [W1] as follows:

“ The weak nuclear force first turned up in the discovery of radioactivity by Henri Becquerel in 1896. In the 1930’s it became understood that in the particular kind of radioactivity that was discovered by Becquerel , known as beta decay, the weak nuclear force causes a neutron inside the nucleus to turn into a proton, at the same time creating an electron and another particle known today as antineutrino, and spitting them out of the nucleus. This is something that is not allowed to happen through any other kind of force. The strong nuclear force that holds the protons and neutrons together in the nucleus and the electromagnetic force that tries to push the

protons in the nucleus apart cannot change the identities of these particles, and the gravitational force certainly does not do anything of the sort, so the observation of neutrons changing into protons or protons into neutrons provided evidence of a new kind of force in the nature. As its name implies, the weak nuclear force is weaker than the electromagnetic or the strong nuclear forces. This is shown for instance by the fact that nuclear beta decay is so slow; the fastest nuclear beta decays take on the average about a hundredth of a second; languorously slow compared with the typical time scale of processes caused by the strong nuclear force, which is roughly a millionth millionth millionth millionth of a second.

In 1933 Enrico Fermi took the first significant step toward a theory of this new force. ... There followed a quarter century of experimental effort aimed at tying up the loose ends of the Fermi theory. ... In 1957 this [problem] was settled and the Fermi theory of the weak nuclear force was put into its final form. ... Nevertheless, even though we had a theory that was capable of accounting for everything that was known experimentally about the weak force, physicists in general found the theory highly unsatisfactory.... The things that were wrong with the Fermi theory were not experimental but theoretical. ... when the theory was applied to more exotic processes it gave nonsensical results....when they did the calculations the answer would turn out to be infinite.... Infinities like these had been encountered in the theory of electromagnetic forces by Oppenheimer and others in the early 1930's, but in the late 1940's theorists had found that all these infinities in quantum electrodynamics would cancel when the mass and electric charge of the electron are properly defined, or "renormalized". As more and more became known about the weak forces it became increasingly clear that the infinities in Fermi's theory of the weak forces would not cancel in this way; the theory was not renormalizable. The other thing that was wrong with the theory of weak forces was that it has a large number of arbitrary elements....

I had worked on the theory of weak forces off and on since graduate school, but in 1967 I was working instead on the strong nuclear forces, the forces that hold neutrons and protons together inside atomic nuclei. I was trying to develop a theory of the strong nuclear forces based on an analogy with quantum electrodynamics. I thought that the difference between the strong nuclear forces and electromagnetism might be explained by a phenomenon known as broken symmetry, which I explain later. It did not work. I found myself developing a theory that did not look like at all the strong forces as they were known to us experimentally. Then it suddenly occurred to me that these ideas, although they had turned out to be completely useless as far as the strong forces were concerned, provided a mathematical basis for a theory of weak nuclear forces that would do anything that one might

want. I could see the possibility of a theory of the weak force analogous to quantum electrodynamics. Just as the electromagnetic force between distant charged particles is caused by the exchange of photons, a weak force would not act all at once at a single point in space (as in the Fermi theory) but it would be caused by the exchange of photonlike particles between particles at different positions. These new photonlike particles could not be massless like the photon (for one thing, if massless they would have been discovered long before), but they were introduced into the theory in a way that was so similar to the way that the photon appears in quantum electrodynamics that I thought that the theory might be renormalizable in the same sense as quantum electrodynamics—that is, that the infinities in the theory could be canceled by a redefinition of the masses and other quantities in the theory. Furthermore, the theory would be highly constrained by its underlying principles and would thus avoid a large part of arbitrariness of previous theories.

I worked out a particular concrete realization of this theory, that is, a particular set of equations that govern the way the particles interacted and that would have the Fermi theory as a low energy approximation. I found in doing this, although it had not been my idea at all to start with, that it turned out to be a theory not only of the weak forces, based on an analogy with electromagnetism; it turned out to be a unified theory of the weak and electromagnetic forces that showed that they were both just different aspects of what subsequently became called an electroweak force. The photon, the fundamental particle whose emission and absorption causes electromagnetic forces, was joint in a tight-knit family group with the other photonlike particles predicted by the theory: electrically charged W particles whose exchange produces the weak force of beta radioactivity, and a neutral particle I called the “ Z ”, about which more later. (W particles were an old story in speculations about the weak forces; the W stands for “weak”. I picked the letter Z for their new sibling because the particle has zero electric charge and also because Z is the last letter of the alphabet, and I hoped that this would be the last member of the family.) Essentially the same theory was worked out independently in 1968 by the Pakistani physicist Abdus Salam, working in Trieste.... Both Salam and I had stated our opinion that this theory would eliminate the problem of infinities in the weak forces. But we were not clever enough to prove this. In 1971 I received a preprint from a young graduate student at the University of Utrecht named Gerard 't Hooft, in which he claimed to show that this theory actually had solved the problem of the infinities: the infinities in calculations of observable quantities would in fact all cancel just just as in quantum electrodynamics....”

7 Unified wave mechanics.

There are two natural ways to furnish the time on nilpotent groups. One of them is defined by solvable extensions while in the other case the time-axis is introduced by Cartesian product with the real line \mathbb{R} . These two extensions are called expanding- and static-models respectively.

The metric on the nilpotent group is positive definite where the Laplacian turns out to be natural physical Hamilton operator corresponding to elementary particle systems. Concrete systems are represented by the corresponding invariant subspaces of the Laplacian and one obtains the Hamilton operator of a given system by restricting the Laplacian onto these subspaces. In order to establish the wave equations regarding these Hamiltonians, on both extensions indefinite metrics must be introduced. That is, adequate wave mechanics associated with these Hamiltonians can just relativistically be introduced such that one assumes appropriate Lorenz-indefinite metrics on both extensions. As it turns out, these metrics really provide the familiar wave operators of wave mechanics.

7.1 Solvable extensions.

Any 2-step nilpotent Lie group, N , extends to a solvable group, SN , which is defined on the half-space $\mathcal{N} \times \mathbb{R}_+$ by the group multiplication

$$(101) \quad (X, Z, t)(X', Z', t') = (X + t^{\frac{1}{2}}X', Z + tZ' + \frac{1}{2}t^{\frac{1}{2}}[X, X'], tt').$$

This formula provides the multiplication also on the nilpotent group N , which appears as a subgroup on the level set $t = 1$.

The Lie algebra of this solvable group is $\mathcal{S} = \mathcal{N} \oplus \mathcal{T}$. The Lie bracket is completely determined by the formulas

$$(102) \quad [\partial_t, X] = \frac{1}{2}X \quad ; \quad [\partial_t, Z] = Z \quad ; \quad [\mathcal{N}, \mathcal{N}]_{SN} = [\mathcal{N}, \mathcal{N}]_N,$$

where $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$.

The indefinite metric tensor is defined by the left-invariant extension of the indefinite inner product, \langle, \rangle , defined on the solvable Lie algebra \mathcal{S} by $\langle \partial_t, \partial_t \rangle = -1$, $\langle \partial_t, \mathcal{N} \rangle = 0$, and $\langle \mathcal{N}, \mathcal{N} \rangle = \langle \mathcal{N}, \mathcal{N} \rangle_{\mathcal{N}}$. The last formula indicates that the original innerproduct is kept on the subalgebra \mathcal{N} . Lie algebra \mathcal{S} is considered as the tangent space at the origin $(0, 0, 1) \in SN$ of the solvable group and \langle, \rangle is extended to a left-invariant metric g onto SN by the group multiplication described above. Scaled inner product \langle, \rangle_q

with scaling factor $q > 0$ is also defined by $\langle \partial_t, \partial_t \rangle_q = -1/q^2$, but, keeping both the inner product on \mathcal{N} and the relation $\partial_t \perp \mathcal{N}$ on \mathcal{N} intact. That is, the scaling regards just the direction regarding ∂_t . The left invariant extension of these inner products are denoted by g_q .

For precise explanations we need some explicit formulas on these groups as well. The left-invariant extensions, $\mathbf{Y}_i, \mathbf{V}_\alpha, \mathbf{T}$, of the unit vectors

$$(103) \quad E_i = \partial_i \quad , \quad e_\alpha = \partial_\alpha \quad , \quad \epsilon = q\partial_t$$

picked up at the origin $(0, 0, 1)$ are the vector fields:

$$(104) \quad \mathbf{Y}_i = t^{\frac{1}{2}} \mathbf{X}_i \quad ; \quad \mathbf{V}_\alpha = t \mathbf{Z}_\alpha \quad ; \quad \mathbf{T} = qt\partial_t,$$

where \mathbf{X}_i and \mathbf{Z}_α are the invariant vector fields introduced on N previously.

One can establish these formulas by the following standard computations. Consider the vectors $\partial_i, \partial_\alpha$ and ∂_t at the origin $(0, 0, 1)$ such that they are the tangent vectors of the curves $c_A(s) = (0, 0, 1) + s\partial_A$, where $A = i, \alpha, t$. Then transform these curves to an arbitrary point by the left multiplications. Then the tangent of the transformed curve gives the desired left invariant vector at an arbitrary point.

According to these formulas, not t but T defined by $\partial_T = \mathbf{T}$ is the correct physical time parameterization on the t - parameter line, which, by the below arguments, are geodesics on SN . The transformation law $\partial_T = (dt/dT)\partial_t$ yields the relations $(dt/dT) = qt$, $\ln t = qT$, and $t = e^{qT}$. Thus a t -level set is the same as the $T = (\ln t)/q$ -level set and subgroup N corresponds both to $t = 1$ and $T = 0$. The reversed time $-T$ is denoted by τ .

Let $c_x(s)$ and $c_z(s)$ be integral curves of finite length $\|c_x\|$ resp. $\|c_z\|$ of the invariant vector fields \mathbf{X} and \mathbf{Z} on the subgroup N . Then the flow generated by ∂_τ moves these curves to $c_x^\tau(s)$ resp. $c_z^\tau(s)$ of length $\|c_x^\tau\| = \|c_x\|e^{q\tau/2}$ resp. $\|c_z^\tau\| = \|c_z\|e^{q\tau}$. That is, by considering them as functions of the time-variable τ , the length is increasing such that the rate of change (derivative with respect to τ) is proportional to the length of the curves. In other words, this mathematical space-time model represents an expanding micro universe where the distance between particles is growing exactly in the same way how this growing distance was measured by Hubble, in 1929, between galaxies [H].

Edwin Hubble came to this conclusion after experimenting red-shift in the spectra of galaxies he observed for cataloguing their distances from the earth. It was quite a surprise that, contrary to the expectation, the red- and blue-shifted galaxies occurred with no equal likelihood but most of them appeared red-shifted. That is, most of them are moving away from us. Even

more surprising was to find that even the size of a galaxy's red shift is not random, but it is directly proportional to the galaxy's distance from us. This means, that the farther the galaxy is, the faster it is moving away. This is the familiar Hubble's law which was actually predicted by the Friedmann cosmological model, in 1922.

The tendency to expand must be rooted from the very same tendency inbuilt into the microscopic universe. This argument, however, contradicts the experimental fact (explained in the introduction) according to which the particles are not expanding. This paradox can be resolved, however, by recognizing that, due to the expansion, the change of the constant magnetic fields, which are present in the particles also by the new model, induces electromagnetic waves which are immediately radiated out into the space. This explanation clarifies not only this paradox but it casts also a new light to the presence of the constant radiation experimentally known in the space. These arguments greatly enhances the importance of the exact mathematical models introduced in this paper.

The covariant derivative can be computed by the well known formula

$$(105) \quad \langle \nabla_P Q, R \rangle = \frac{1}{2} \{ \langle P, [R, Q] \rangle + \langle Q, [R, P] \rangle + \langle [P, Q], R \rangle \},$$

where P, Q, R are invariant vector fields. Then we get

$$\begin{aligned} \nabla_{X+Z}(X^* + Z^*) &= \nabla_{X+Z}^N(X^* + Z^*) - q\left(\frac{1}{2}\langle X, X^* \rangle + \langle Z, Z^* \rangle\right)\mathbf{T}; \\ \nabla_X \mathbf{T} &= \frac{q}{2}X \quad ; \quad \nabla_Z \mathbf{T} = qZ \quad ; \quad \nabla_T X = \nabla_T Z = \nabla_T T = 0, \end{aligned}$$

where ∇^N denotes covariant derivative and $X, X^* \in \mathcal{X}; Z, Z^* \in \mathcal{Z}; T \in \mathcal{T}$.

The Laplacian on these solvable groups can be established by the same computation performed on N . Then we get

$$\begin{aligned} (106) \quad \Delta &= t^2 \Delta_Z - q^2(t^2 \partial_{tt}^2 + t \partial_t) + \\ &t\left(\Delta_X + \frac{1}{4} \sum_{\alpha, \beta=1}^l \langle J_\alpha(X), J_\beta(X) \rangle \partial_{\alpha\beta}^2 + \sum_{\alpha=1}^l \partial_\alpha D_\alpha \bullet\right) + q^2\left(\frac{k}{2} + l\right)t \partial_t = \\ &= e^{2qT} \Delta_Z - \partial_{TT}^2 + \\ &e^{qT} \left(\Delta_X + \frac{1}{4} \sum_{\alpha, \beta=1}^l \langle J_\alpha(X), J_\beta(X) \rangle \partial_{\alpha\beta}^2 + \sum_{\alpha=1}^l \partial_\alpha D_\alpha \bullet\right) + q\left(\frac{k}{2} + l\right) \partial_T. \end{aligned}$$

This is the Laplacian on the solvable extension of a general 2-step nilpotent Lie group. On the extension of a H-type group it appears in the following

simpler form:

$$(107) \quad \Delta = \{e^{2qT} \Delta_Z - \partial_{TT}^2\} + \{e^{qT} (\Delta_X + \frac{1}{4} \mathbf{x}^2 \Delta_Z + \sum \partial_\alpha D_\alpha \bullet) + q(\frac{k}{2} + l) \partial_T\}$$

This operator is expressed regarding the collapsing time direction. Substitution $T = -\tau$ provides the operator in terms of the expanding time direction. Note that the first operator, $e^{-2q\tau} \Delta_Z - \partial_{\tau\tau}^2$, looks like "expanding meson operator", while the second one is similar to the Schrödinger operator of charged particles. This question is further investigated in the next section.

In order to understand the deeper connections to general relativity, also the Riemannian curvature should explicitly be computed. This calculation can straightforwardly be implemented by substituting formulas (106) into the standard formula of the Riemannian curvature. Then we get

$$(108) \quad R_q(X^* \wedge X) = R(X^* \wedge X) + \frac{q}{2} [X^*, X] \wedge \mathbf{T} + \frac{q^2}{4} X^* \wedge X;$$

$$(109) \quad R_q(X \wedge Z) = R(X \wedge Z) + \frac{q}{4} J_Z(X) \wedge \mathbf{T} + \frac{q^2}{2} X \wedge Z;$$

$$(110) \quad R_q(Z^* \wedge Z) = R(Z^* \wedge Z) + q^2 Z^* \wedge Z;$$

$$(111) \quad R_q((X + Z), \mathbf{T})(\cdot) = q \nabla_{\frac{1}{2}X+Z}(\cdot); \quad R_q((X + Z) \wedge \mathbf{T}) =$$

$$(112) \quad = \frac{1}{2} q \left(\sum_\alpha J_\alpha(X) \wedge e_\alpha - J_Z^* \right) - q^2 \left(\frac{1}{4} X + Z \right) \wedge \mathbf{T},$$

where J_Z^* is the 2-vector dual to the 2-form $\langle J_Z(X_1), X_2 \rangle$ and R is the Riemann curvature on N . The vectors in these formulas are elements of the Lie algebra.

By introducing $H(X, X^*, Z, Z^*) := \langle J_Z(X), J_{Z^*}(X^*) \rangle$, for the Ricci curvature we have

$$(113) \quad Ri_q(X) = Ri(X) - q^2 \left(\frac{k}{4} + \frac{l}{2} \right) X;$$

$$(114) \quad Ri_q(Z) = Ri(Z) - q^2 \left(\frac{k}{2} + l \right) Z \quad ; \quad Ri_q(T) = +q^2 \left(\frac{k}{4} + l \right) T,$$

where the Ricci tensor Ri on N is described by formulas

$$Ri(X, X^*) = -\frac{1}{2} \sum_{\alpha=1}^l H(X, X^*, e_\alpha, e_\alpha) = -\frac{1}{2} H_X(X, X^*) = -\frac{l}{2} \langle X, X^* \rangle;$$

$$Ri(Z, Z^*) = \frac{1}{4} \sum_{i=1}^k H(E_i, E_i, Z, Z^*) = \frac{1}{4} H_Z(Z, Z^*) = \frac{k}{4} \langle Z, Z^* \rangle,$$

and by $Ri(X, Z) = 0$. By assuming $q = 1$, we have

$$(115) \quad Ri_1(X) = -(\frac{k}{4} + l)X; \quad Ri_1(Z) = -(\frac{k}{4} + l)Z;$$

$$(116) \quad Ri_1(T) = (\frac{k}{4} + l)T, \quad \mathcal{R} = -(\frac{k}{4} + l)(k + l + 1),$$

$$(117) \quad Ri_1(X + Z, X^* + Z^*) - \frac{1}{2}\mathcal{R}\langle X + Z, X^* + Z^* \rangle =$$

$$(118) \quad = (\frac{k}{4} + l)(k + l - \frac{1}{2})\langle X + Z, X^* + Z^* \rangle,$$

$$(119) \quad Ri_1(X + Z, T) - \frac{1}{2}\mathcal{R}\langle X + Z, T \rangle = 0,$$

$$(120) \quad Ri_1(T, T) - \frac{1}{2}\mathcal{R}\langle T, T \rangle = (\frac{k}{4} + l)(k + l + \frac{3}{2})\langle T, T \rangle.$$

These tensors are defined in terms of the elements of the Lie algebra. In order to compute them on local coordinate systems, first the metric tensor $g_{ij} = g(\partial_i, \partial_j)$, $g_{i\alpha} = g(\partial_i, \partial_\alpha)$, $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$ and its inverse, g^{ij} , $g^{i\alpha}$, $g^{\alpha\beta}$, on N , need to be calculated. By the explicit form of the invariant vector fields we have:

$$(121) \quad g_{ij} = \delta_{ij} + \frac{1}{4} \sum_{\alpha=1}^l \langle J_\alpha(X), \partial_i \rangle \langle J_\alpha(X), \partial_j \rangle, \quad g_{\alpha\beta} = \delta_{\alpha\beta},$$

$$(122) \quad g_{i\alpha} = -\frac{1}{2} \langle J_\alpha(X), \partial_i \rangle, \quad g^{ij} = \delta_{ij}, \quad g^{i\alpha} = \frac{1}{2} \langle \partial_i, J_\alpha(X) \rangle,$$

$$(123) \quad g^{\alpha\beta} = \delta_{\alpha\beta} + \frac{1}{4} \langle J_\alpha(X), J_\beta(X) \rangle = (1 + \frac{1}{4}\mathbf{x}^2)\delta_{\alpha\beta}.$$

These components determine the metric tensor on SN by the formulas: $tg_{ij} =, t^{3/2}g_{i\alpha}, t^2g_{\alpha\beta}$.

7.2 Static Schrödinger and neutrino equations.

The static model is defined by the Cartesian product, $N \times \mathbb{R}$, of metrics, where \mathbb{R} , parameterized by t , is endowed by the indefinite inner product $\langle \partial_t, \partial_t \rangle_q = -1/q^2$. The several objects such as Riemann curvature can be computed by laws corresponding to the Cartesian products, thus they are non-trivial only regarding the nilpotent direction. These explicit formulas can easily be recover from the previous ones.

In what follows we utilize Pauli's computation (10)-(16) regarding the non-relativistic approximation of the relativistic wave equation. By choosing

$q = 1/c$, the Laplacian appears in the following form:

$$(124) \quad \Delta = (\Delta_Z - \frac{1}{c^2} \partial_{tt}^2) + (\Delta_X + \frac{1}{4} \mathbf{x}^2 \Delta_Z + \sum \partial_\alpha D_\alpha \bullet) = \\ = (\Delta_Z + \frac{2m\mathbf{i}}{\hbar} \partial_t - \frac{1}{c^2} \partial_{tt}^2) + (\Delta_X + \frac{1}{4} \mathbf{x}^2 \Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar} \partial_t).$$

On the Z -space, operator $\Delta_Z - \frac{1}{c^2} \partial_{tt}^2$ is nothing but the wave operator (17). According to Yukawa's exposition, the eigenfunctions, U , of this operator describe the eigenstates of nuclear forces. Due to (7), the general solutions of this wave equation are de Broglie's wave packets (6). On the mathematical model, however, these wave packets are represented by twisted functions of the form

$$(125) \quad \psi_{\mathbf{B}p_i q_i}(X, Z, t) =$$

$$(126) \quad = \int_{\mathbb{R}^l} e^{\mathbf{i}(\langle Z, K \rangle - \omega t)} \phi(\mathbf{x}, \mathbf{k}) \varphi(K_u) \prod_{i=1}^{k/2} z_{K_u i}^{p_i}(X) \bar{z}_{K_u i}^{q_i}(X) dK =$$

$$(127) \quad = \int_{\mathbb{R}^l} e^{\mathbf{i}(\langle Z, K \rangle - \omega t)} \phi(\mathbf{x}, \mathbf{k}) F_{\mathbf{B}p_i q_i}(X, K_u) dK,$$

$$(128) \quad \text{where} \quad \sqrt{\mathbf{k}^2 + \frac{m^2 c^2}{\hbar^2}} = \frac{\omega}{c}.$$

Wave packets $\psi_{Qpq}(X, Z, t)$ are defined by means of the functions

$$(129) \quad F_{Qpq}(X, K_u) = \varphi(K_u) (\Theta_Q^p \bar{\Theta}_Q^q)(X, K_u).$$

They are defined also for Z -sphere bundles $S_R(\mathbf{x})$, which versions are indicated by denotations $\psi_{\mathbf{B}p_i q_i S_R}$ and $\psi_{Qpq S_R}$. From this respect, $\psi_{\mathbf{B}p_i q_i \mathbb{R}^l}$ and $\psi_{Qpq \mathbb{R}^l}$ correspond to the above introduced wave packets.

For a fixed Z -vector, Z_γ , the regarding denotations are $\psi_{\mathbf{B}p_i q_i Z_\gamma}$ and $\psi_{Qpq Z_\gamma}$, where \mathbf{B} is an orthonormal basis regarding $J_{Z_\gamma u}$. In this case the integral is taken with respect to the Dirac delta measure concentrated at Z_γ , thus these formulas can be written up without indicating this integral or constant $\varphi(K_u)$. By projections $\Pi_X^{(n)}$, one defines

$$(130) \quad \Psi_{\dots}(X, Z, t) = \int_{\circ} e^{\mathbf{i}(\langle Z, K \rangle - \omega t)} \phi(\mathbf{x}, \mathbf{k}) \Pi_X^{(n)} F_{\dots}(X, K_u) dK,$$

where dots can be substituted by any of the symbols $\mathbf{B}p_i q_i \mathbb{R}^l$, $Qpq \mathbb{R}^l$, .. e. t. c., and circle, \circ , could symbolize any of the integral domains \mathbb{R}^l , S_R , Z_γ .

If also projections $\Pi_K^{(r,s)}$ are applied to F_{\dots} , the corresponding functions are $\psi_{\dots}^{(r,s)}$ resp. $\Psi_{\dots}^{(r,s)}$. This operation makes sense only for integral domains

\mathbb{R}^l or $S_R(\mathbf{x})$ but it is not defined for the singular Dirac delta domain Z_γ . Anti de Broglie wave packets are defined by replacing $-\omega$ with $+\omega$ in the above formulas. The corresponding functions are denoted by ψ_{\dots}^{anti} and Ψ_{\dots}^{anti} . The associated particles are called antiparticles. These objects can be introduced also by keeping $-\omega$ and replacing \mathbf{i} by $-\mathbf{i}$.

As it is indicated, the right side of (124) is computed by adding $\frac{2m\mathbf{i}}{\hbar}\partial_t - \frac{2m\mathbf{i}}{\hbar}\partial_t = 0$ to the left side. Then the wave operator associated with nuclear forces becomes

$$(131) \quad N = \Delta_Z + \frac{2m\mathbf{i}}{\hbar}\partial_t - \frac{1}{c^2}\partial_{tt}^2,$$

which is the non-relativistic wave operator established in (16). As it is explained in (12)-(16), the N -harmonic waves, defined by $N(\tilde{\psi})(Z, t) = 0$, relate to the relativistic wave by the formula

$$(132) \quad \psi(Z, t) = e^{-\frac{imc^2}{\hbar}t}\tilde{\psi}(Z, t).$$

Also remember that frequency $\tilde{\omega}$ is derived from

$$(133) \quad \omega = \frac{E}{\hbar} = \frac{mc^2}{\hbar} \sqrt{1 + \frac{\hbar^2 \mathbf{k}^2}{m^2 c^2}} = \frac{mc^2}{\hbar} + \tilde{\omega} = \frac{mc^2}{\hbar} + \frac{\hbar}{2m} \mathbf{k}^2 + \dots,$$

by the Taylor expansion of function $\sqrt{1+x}$. Thus the third term depends on \hbar^3 and, by stepping further, this exponent is increased by 2, by each step. For low speed particles, value $\tilde{\omega} = \frac{\hbar}{2m} \mathbf{k}^2$ is a good approximation of the frequency, thus also $E = \hbar\tilde{\omega}$ is a good approximation for the energy of the particle associated with this non-relativistic wave. Note that $E = E_{kin}$ is nothing but the kinetic energy owned by the material particle. By this reason, the particle associated with the wave operator N can be consider as one of the residues of a decaying material particle which has neither mass nor charge and the only source of its energy is the kinetic energy of the decaying material particle. Such particles are the neutrinos, thus N is called neutrino operator accompanying the electron-positron system.

The energy mc^2 of the material particle is completely attributed to the other particle associated with the second operator

$$(134) \quad S = \Delta_X + \frac{1}{4}\mathbf{x}^2\Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t.$$

incorporated into the Laplacian (124). In order to understand this particle represented by this operator, we introduce first the de Broglie wave packets $\tilde{\Psi}_{\dots}(X, Z, t)$ and $\tilde{\Psi}_{\dots}^{anti}(X, Z, t)$ in the same way as before, but now, the ω is replaced by $\tilde{\omega}$ which can take values such as $\frac{\hbar}{2m} \mathbf{k}^2$, $\frac{\hbar}{2m}(4r+4p+k)\mu$, $\frac{\hbar}{2m}((4r+4p+k)\mu + 4\mu^2)$. By (24), the Schrödinger equation for an electron is:

$$(135) \quad -(\Delta_{(x,y)} - \frac{eB}{\hbar c} D_z \bullet + \frac{e^2 B^2}{4\hbar^2 c^2} (x^2 + y^2))\psi = \frac{2m\mathbf{i}}{\hbar} \frac{\partial \psi}{\partial t}.$$

On the Z-crystal model, operator \triangleleft_μ is defined by the action of the Laplacian Δ of the nilpotent group on functions of the form $\psi(X)e^{2\pi\mathbf{i}\langle Z_\gamma, Z \rangle}$. In terms of $\lambda = \pi\mathbf{z}_\gamma$, this action can be described as acting only on ψ by the operator $\triangleleft_\mu = \Delta_X + 2\mathbf{i}D_\mu \bullet - \mu^2 \mathbf{x}^2 - 4\mu^2$. If the last constant term is omitted, the operator left is denoted by \sqcup_μ . Then, in the 2D case after substitution $\mu = eB/2\hbar c$, the negative of this reduced operator becomes nothing but the Hamilton operator standing on the left side of the above Schrödinger equation. If $\tilde{K} = 2\pi Z_\gamma$, $\mu = \tilde{\mathbf{k}}/2$, $m = \kappa m_e$, and $f_\mu(\mathbf{x}^2)$ is a function such that $f_\mu(\tilde{t})$ is an eigenfunction of the radial Landau-Zeeman operator $\diamond_{\tilde{t}} + 4\mu^2$, defined in (55), with eigenvalue $-\tilde{\omega} = -(4r + 4p + k)\mu$, then for $S(\tilde{\Psi}_{\dots\tilde{K}}^{anti}(X, Z, t))$ we have:

$$(136) \quad \begin{aligned} & S(e^{\mathbf{i}(\langle Z, \tilde{K} \rangle + \frac{\hbar}{2m}\tilde{\omega}t)} f_\mu(\mathbf{x}^2) \Pi_X^{(n)} F_{\dots}(X, \tilde{K}_u)) = \\ & = e^{\mathbf{i}\langle Z, \tilde{K} \rangle} (\sqcup_\mu - \frac{2m\mathbf{i}}{\hbar} \frac{\partial}{\partial t}) (e^{\frac{\hbar\mathbf{i}}{2m}\tilde{\omega}t} f_\mu(\mathbf{x}^2) \Pi_X^{(n)} F_{\dots}(X, \tilde{K}_u)) = 0. \end{aligned}$$

Thus on Z-crystals, operator S is nothing but Schrödinger's classical wave operator of an electron positron system.

Note that no non-relativistic limiting was used to obtain this operator. It is naturally incorporated into the complete Laplacian Δ . Although it is the same as the non-relativistic wave operator obtained earlier by non-relativistic limiting, even the neutrino operator, N , is not the result of a non-relativistic limiting. The Laplacian Δ is the sum of these two natural operators, meaning that it actually represents a system consisting of electrons positrons and electron-positron-neutrinos. The above arguments also suggest that this system can be regarded as the result of a sort of nucleus-decay. A rigorous theory describing this process is yet to be established. It is clear, however, that the basic mathematical tool underlying this physical theory must be the decomposition of the Laplacian into operators corresponding to the constituents of a given particle system. Dealing with Laplacian means that one does not violate the principle of energy conservation. Moreover, this tool provides also the exact operators associated with the particles, which is the most attractive new feature of these exact mathematical models.

Actually, the elementary particles discovered in classical quantum theory were introduced by the very same idea. For instance, the neutrino was first postulated in 1930 by Wolfgang Pauli to preserve conservation of energy, conservation of momentum, and conservation of angular momentum in beta

decay the decay of a neutron into a proton, an electron and an antineutrino. Pauli theorized that an undetected particle was carrying away the observed difference between the energy, momentum, and angular momentum of the initial and final particles. The only difference between the two ways introducing the neutrinos is that Pauli did not have a Riemann manifold in hand in the Laplacian of which he would have been able to separate the neutrino from the other particles resulted by the decay.

The only term in the Laplacian containing second order derivatives regarding the time variable t is the neutrino operator. This term is of first order in the Schrödinger operator. Because of this, waves $\tilde{\Psi}$ are not solutions of the neutrino operator and waves $\tilde{\psi}$ obtained above by the Taylor expansion are not solutions of the Schrödinger equation. In order to cope with this difficulty, non-relativistic approximation can be implemented such that one attributes the kinetic energy represented by Δ_Z in the neutrino operator to the Hamilton operator associated with S by considering the total Schrödinger operator

$$(137) \quad \mathbb{S} = \Delta_X + (1 + \frac{1}{4}\mathbf{x}^2)\Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t,$$

which is the sum of the Schrödinger operator and Δ_Z .

In this step, the two operator is pulled together to form an operator which is of first order regarding the time variable. This scheme is completely analogous to those applied by Schrödinger when, instead of the Klein-Gordon equation, he introduced his equation. A major difference is, however, that the above operator accounts also with the energy of neutrinos accompanying the electron-positron system, moreover, the non-relativistic approximation is applied to the neutrino operator and not to the electron-positron operator. The neglected Taylor-terms in this approximation depend on \hbar^s , where $s \geq 3$. The wave functions regarding this pulled-together operator are defined by the eigenvalues $-\tilde{\omega} = -((4r + 4p + k)\mu + 4\mu^2)$. Then, in terms of $\mathbf{F}_{\dots}^{(n)}(X, \tilde{K}_u) = \Pi_X^{(n)} F_{\dots}(X, \tilde{K}_u)$, we have:

$$(138) \quad \mathbb{S}(\tilde{\Psi}_{\dots\tilde{K}}^{anti}(X, Z, t)) = \mathbb{S}(e^{i(\langle Z, \tilde{K} \rangle + \frac{\hbar}{2m}\tilde{\omega}t)} f_\mu(\mathbf{x}^2) \mathbf{F}_{\dots}^{(n)}(X, \tilde{K}_u)) = \\ = e^{i\langle Z, \tilde{K} \rangle} (\triangleleft_\lambda - \frac{2m\mathbf{i}}{\hbar} \frac{\partial}{\partial t}) (e^{\frac{\hbar\mathbf{i}}{2m}\tilde{\omega}t} f_\mu(\mathbf{x}^2) \mathbf{F}_{\dots}^{(n)}(X, \tilde{K}_u)) = 0.$$

$$(139) \quad \mathbb{S}(\tilde{\Psi}_{\dots S_R}^{anti}) = \mathbb{S}(\oint_{S_R} e^{i(\langle Z, K \rangle + \frac{\hbar}{2m}\tilde{\omega}t)} f_{\frac{1}{2}\mathbf{k}}(\mathbf{x}^2) \mathbf{F}_{\dots}^{(n)}(X, K) dK_n) = \\ = \oint_{S_R} e^{i\langle Z, K \rangle} (\triangleleft_{\frac{1}{2}\mathbf{k}} - \frac{2m\mathbf{i}}{\hbar} \frac{\partial}{\partial t}) (e^{\frac{\hbar\mathbf{i}}{2m}\tilde{\omega}t} f_{\frac{1}{2}\mathbf{k}}(\mathbf{x}^2) \mathbf{F}_{\dots}^{(n)}(X, K) dK_n) = 0.$$

$$(140) \quad \mathbb{S}(\tilde{\Psi}_{\dots\mathbb{R}^l}^{anti}) = \mathbb{S}\left(\int_{\mathbb{R}^l} e^{i(\langle Z, K \rangle + \frac{\hbar}{2m}\tilde{\omega}t)} f_{\frac{1}{2}\mathbf{k}}(\mathbf{x}^2) \mathbf{F}_{\dots}^{(n)}(X, K_u) dK\right) = \\ \int e^{i\langle Z, K \rangle} \left(\triangleleft_{\frac{1}{2}\mathbf{k}} - \frac{2m\mathbf{i}}{\hbar} \frac{\partial}{\partial t}\right) \left(e^{\frac{\hbar\mathbf{i}}{2m}\tilde{\omega}t} f_{\frac{1}{2}\mathbf{k}}(\mathbf{x}^2) \mathbf{F}_{\dots}^{(n)}(X, K_u)\right) \mathbf{k}^{l-1} dK_u d\mathbf{k} = 0.$$

The same formulas hold for operator S which appears as the classical Schrödinger operator $\triangleleft_{\frac{1}{2}\mathbf{k}} - \frac{2m\mathbf{i}}{\hbar} \frac{\partial}{\partial t}$ behind the integral sign. Similar arguments work out also for operator \mathbb{E} , in which case the Schrödinger operators act on wave functions $\tilde{\Psi}_{\dots S_R}^{anti(r,s)}(X, Z, t)$. This is still a scalar operator which can be reduced to a radial operator acting on a single radial function. The radial operator to which the complete operator Δ can be reduced act on d-tuples of radial function, therefore integral formulas regarding these cases must be built up in terms of function $f_\beta \Pi_{K_u}^{(\beta)} \mathbf{F}^{(n)}$ where d-tuple (f_1, \dots, f_d) is an eigen d-tuple of the reduced radial operator. The particles defined by these operators are denoted by $W_{\mathbb{E}} = W_{\mathbb{E}(1)}$ resp. $W_{\Delta} = W_{\Delta}^{(\infty)}$. They are called clean-weak and clean-high W-particles respectively, while the other particles $W_{\mathbb{E}}^{(u)} = W_{\mathbb{E}(u)}$ resp. $W_{\Delta}^{(u)} = W_{\Delta(u)}$ are the so called dirty W-particles. The neutrino operator is the same in all of these cases, thus the associated particles are denoted $Z_{\mathbb{E}}$. These denotations are suggested by the theory of weak nuclear forces. They indicate that W-type particles can analogously be defined also regarding strong forces. However, the beta decay can be explained just by the clean weak nuclear forces.

7.3 Expanding Schrödinger and neutrino equations.

For the sake of simplicity the following formulas are established regarding the collapsing (shrinking) time-direction T under the condition $q = 1$. Formulas regarding the expanding time-direction τ can be obtained by the substitution $T = -\tau$. Instead of t , the expanding wave functions are introduced in terms of e^T . That is, the shrinking twisted wave packets are of the form

$$(141) \quad \Psi_{\dots}(X, Z, T) = \int_{\circ} e^{i(\langle Z, K \rangle - \omega e^T)} \phi(\mathbf{x}, \mathbf{k}) \Pi_X^{(n)} F_{\dots}(X, K_u) dK \\ = \int_{\circ} e^{i(\langle Z, K \rangle - \omega e^T)} \phi(\mathbf{x}, \mathbf{k}) \mathbf{F}_{\dots}^{(n)}(X, K_u) dK,$$

where $\sqrt{\mathbf{k}^2 + \frac{m^2 c^2}{\hbar^2}} = \frac{\omega}{c}$, and, as above, dots can be substituted by any of the symbols $\mathbf{B}p_i q_i \mathbb{R}^l$, $Qp q \mathbb{R}^l$, .. e. t. c., and circle, \circ , could symbolize any of the integral domains \mathbb{R}^l , S_R , Z_γ . De Broglie's wave packets $\tilde{\Psi}_{\dots}(X, Z, T)$ and $\tilde{\Psi}_{\dots}^{anti}(X, Z, T)$ are introduced by the same modification, that is, the

ω is replaced by $\tilde{\omega}$ in the latter formula, which can take values such as $\frac{\hbar}{2m}\mathbf{k}^2, \frac{\hbar}{2m}(4r+4p+k)\mu, \frac{\hbar}{2m}((4r+4p+k)\mu+4\mu^2)$.

The meson operator appears now in the form:

$$(142) \quad M = e^{2T} \Delta_Z + \partial_T - \partial_{TT}^2,$$

The same computation implemented on the static model shows that the shrinking matter waves $\Psi_{\dots}(X, Z, T)$, defined in terms of ω , are really harmonic, meaning $M\Psi_{\dots} = 0$, regarding this operator. Moreover, wave packet $\hat{\Psi}(X, Z, T)$ defined by

$$(143) \quad \Psi(X, Z, T) = e^{-\frac{imc^2}{\hbar}e^T} \hat{\Psi}(X, Z, T)$$

is harmonic regarding the shrinking neutrino operator

$$(144) \quad N = e^{2T} \Delta_Z + (1 + \frac{2m\mathbf{i}}{\hbar}e^T)\partial_T - \partial_{TT}^2.$$

According to this computation, the corresponding decomposition of the Laplacian into non-polarized neutrino and Schrödinger operator of a particle system is as follows

$$(145) \quad \Delta = \{e^{2T} \Delta_Z - \partial_{TT}^2\} +$$

$$(146) \quad + \{e^T (\Delta_X + \frac{1}{4}\mathbf{x}^2 \Delta_Z + \sum \partial_\alpha D_\alpha \bullet) + (\frac{k}{2} + l)\partial_T\} =$$

$$= \{e^{2T} \Delta_Z + (1 + \frac{2m\mathbf{i}}{\hbar}e^T)\partial_T - \partial_{TT}^2\} +$$

$$(147) \quad + \{e^T (\Delta_X + \frac{1}{4}\mathbf{x}^2 \Delta_Z + \sum \partial_\alpha D_\alpha \bullet) + (\frac{k}{2} + l - 1 - \frac{2m\mathbf{i}}{\hbar}e^T)\partial_T\}$$

$$= (e^{2T} \Delta_Z + (1 + \frac{2m\mathbf{i}}{\hbar}e^T)\partial_T - \partial_{TT}^2) +$$

$$+ e^T (\Delta_X + \frac{1}{4}\mathbf{x}^2 \Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_T) + (\frac{k}{2} + l - 1)\partial_T.$$

In terms of $\tau = -T$, these operators define the expanding non-polarized neutrino, Schrödinger, and tractor operators respectively. The force associated with the third operator supplies the energy what is needed to maintain the expansion. Let it also be pointed out that according to these models the particles are not just moving away from each other but this movement is also accelerating. This acceleration can be computed by taking the second derivatives of the distance function introduced at explaining the expansion. This acceleration can be explained just by this new force represented by the third operator.

It is important to keep in mind that these operators are non-polarized. The polarized operators appear behind the integral sign after these non-polarized operators are acting on the de Broglie waves expressed by means of twisted Z-Fourier transforms.

8 Spectral isotropy.

Spectral isotropy means that, on an arbitrary ball \times ball- or sphere \times ball-type manifold with a fixed boundary condition, for any two unit X -vectors Q and \tilde{Q} , the Laplacian is isospectral on the invariant function spaces $\Xi_Q = \sum_n \Xi_Q^{(n)}$ and $\Xi_{\tilde{Q}} = \sum_n \Xi_{\tilde{Q}}^{(n)}$ satisfying the given boundary condition. Recall that total space Ξ_Q is everywhere dense in the straight space spanned by functions of the form $f(|X|, Z)\langle Q, X \rangle$, furthermore, the boundary conditions can totally be controlled by (X, Z) -radial functions, therefore, this total function space is the same than what is defined in terms of the straight functions.

Next we prove that any of the Heisenberg type groups is spectrally isotropic. On general 2-step nilpotent Lie groups, where the endomorphisms J_Z can have distinct eigenvalues, this statement can be established just in a much weaker form not discussed in this paper. Contrary to these general cases, the H-type groups have the distinguishing characteristics that they represent systems consisting identical particles and their anti-particles. Also note that on the expanding model this spectral isotropy explains why the radiation induced by the change of the constant magnetic field attached to the the spin operator is the same whichever direction it is measured from. This radiation isotropy, which has been measured with great accuracy, actually indicates that the Heisenberg type groups are enough to describe the elementary particles and there is no need to involve more general 2-step nilpotent Lie groups to this new theory.

This spectral isotropy is established by the intertwining operator $\omega_{Q\tilde{Q}pq\bullet} : \Xi_{Qpq\bullet} \rightarrow \Xi_{\tilde{Q}pq\bullet}$, defined by

$$(148) \quad \begin{aligned} \mathcal{H}\mathcal{F}_{Qpq\bullet}(\phi) &= \int_{\bullet} e^{i\langle Z, K \rangle} \phi(\mathbf{x}, K) \Pi_X^{(n)}(\Theta_Q^p \bar{\Theta}_Q^q)(X, K_u) dK_{\bullet} \rightarrow \\ &\rightarrow \mathcal{H}\mathcal{F}_{\tilde{Q}pq\bullet}(\phi) = \int_{\bullet} e^{i\langle Z, K \rangle} \phi(\mathbf{x}, K) \Pi_X^{(n)}(\Theta_{\tilde{Q}}^p \bar{\Theta}_{\tilde{Q}}^q)(X, K_u) dK_{\bullet}, \end{aligned}$$

where heavy dot \bullet may represent $R_Z(\mathbf{x})$ or \mathbb{R}^l . They indicate the function spaces this operator is defined for. The very same operators are defined by corresponding Hankel functions obtained by the Hankel decomposition of the

above functions to each other. This statement immediately follows from the fact that the sums of the Hankel components restore the original functions in the above formulas. Also note that the (X, Z) -radial Hankel functions are the same regarding the two corresponded functions and they differ from each other just by the Hankel polynomials obtained by the projections. Thus the operator defined by Hankel decomposition must really be the same as the above operator. It follows that operator (148) preserves the Hankel decompositions.

An other remarkable property of this transform is that it can be induced by point transformations of the form $(O_{\tilde{Q}Q}, id_Z)$, where id_Z is the identity map on the Z -space and $O_{\tilde{Q}Q}$ is orthogonal transformation on the X -space, transforming subspace $S_{\tilde{Q}}$, spanned by \tilde{Q} and all $J_{Z_u}(\tilde{Q})$, onto the similarly defined S_Q . This part of the map is uniquely determined, whereas, between the complement X -spaces it can be arbitrary orthogonal transformation. One should keep in mind that such a point transformation pulls back a function just from $\Xi_{Q\bullet}$ into the function space $\Xi_{\tilde{Q}\bullet}$ and it is not defined for the whole L^2 function spaces.

Next we prove that this operator intertwines the restrictions of the Laplacian to these invariant subspaces, term by term. Since X -radial functions are mapped to the same X -radial functions, furthermore, also X -spherical harmonics of the same degree are intertwined with each other, the statement holds for Δ_X . This part of the statement can be settled also by the formula $\Delta_X = \sum \partial_{z_{K_u i}} \partial_{\bar{z}_{K_u i}}$ written up in a coordinate system established by an orthonormal complex basis where Q is the first element of this basis $\{Q_{K_u i}\}$, for all K_u . That is, $\Theta_Q = z_1$ holds and the statement really follows by the above formula. A third proof can be derived from the fact that this operator is induced by the above described point transformation.

Due to the relations

$$(149) \quad \begin{aligned} \mathbf{M}\mathcal{F}_{Qpq}(\phi) &= \mathcal{F}_{Qpq}((q-p)\mathbf{k}\phi), \quad \Delta_Z \mathcal{F}_{Qpq}(\phi) = \mathcal{F}_{Qpq}(-\mathbf{k}^2\phi), \\ \mathbf{M}\mathcal{H}\mathcal{F}_{Qpq}(\phi) &= \mathcal{H}\mathcal{F}_{Qpq}((q-p)\mathbf{k}\phi), \quad \Delta_Z \mathcal{H}\mathcal{F}_{Qpq}(\phi) = \mathcal{H}\mathcal{F}_{Qpq}(-\mathbf{k}^2\phi), \end{aligned}$$

the other parts of the Laplacians are also obviously intertwined. In these formulas, the second line follows from the first one by the commutativity of operator $D_{K\bullet}$ with the projection Π_X .

The intertwining property regarding the Dirichlet or Z -Neumann conditions on ball \times ball- resp. sphere \times ball-type domains can also be easily established either by the above point transformations, or with the help of the Hankel transform implying that functions of the form $f(|X|, |Z|)$ appearing

in the transform are intertwined with themselves. Since the boundary conditions are expressed in terms of these double radial functions, this argument provides a second proof for the statement. A third proof can be obtained by the explicit formulas established for twisted functions satisfying these boundary conditions.

The most interesting new feature of this spectral isotropy is that it holds even in cases when the space is not spatially isotropic. Let it be recalled that spatial-isotropy is the first assumption on the Friedmann model and the overwhelming evidence supporting this assumption was exactly the isotropic radiation measured by Penzias and Wilson, in 1965. The mathematical models demonstrate, however, that the spectral isotropy manifests itself even in much more general situations when the space is rather not spatially isotropic. In order to explain this situation more clearly, we describe the isometries of H-type groups in more details.

Generically speaking, these groups are not isotropic regarding the X-space. They satisfy this property just in very rare occasions. Starting with Heisenberg-type groups $H_3^{(a,b)}$, there is a subgroup, $\mathbf{Sp}(a) \times \mathbf{Sp}(b)$, of isometries which acts as the identity on the Z-space and which acts transitively just on the X-sphere of $H_3^{(a+b,0)}$. In this case the intertwining property for operators $\omega_{Q\tilde{Q},\bullet}$ also follows from the existence of isometries transforming Q to \tilde{Q} . But the isometries are not transitive on the X-spheres of the other spaces satisfying $ab \neq 0$. This statement follows from the fact that the complete group of isometries is $(\mathbf{Sp}(a) \times \mathbf{Sp}(b)) \cdot SO(3)$, where the action of $SO(3)$, described in terms of unit quaternions q by

$$\alpha_q(X_1, \dots, X_{a+b}, Z) = (qX_1\bar{q}, \dots, qX_{a+b}\bar{q}, qZ\bar{q}),$$

is transitive on the Z-sphere. Thus the above tool is not available to prove the spectral isotropy in these cases. Yet, by the above arguments, the $\omega_{Q\tilde{Q},\bullet}$ is an intertwining operator on its own right, without the help of the isometries.

Note that the members of a family defined by a fixed number $a + b$ have the same X-space but non-isomorphic isometry groups having different dimensions in general. More precisely, two members, $H_3^{(a,b)}$ and $H_3^{(a',b')}$, are isometrically isomorphic if and only if $(a, b) = (a', b')$ holds up to an order. Furthermore, the sphere×sphere-type manifolds are homogeneous just on $H_3^{(a+b,0)} \simeq H_3^{(0,a+b)}$, while they are locally inhomogeneous, even on the X-spheres, on the other members of the family. Let it be emphasized again that this homogeneity concerns not just the homogeneity of the X-spheres but the whole sphere×sphere-type manifold.

The isometries are well known also for all H-type groups $H_l^{(a,b)}$. The X-space-isotropy is obviously true also on the Heisenberg groups $H_1^{(a,b)}$ which can be defined as H-type groups satisfying $l = 1$. Besides this and the above quaternionic examples, it yet holds just on $H_7^{(1,0)} \simeq H_7^{(0,1)}$. Thus the X-isotropy regarding isometries is a rare property, indeed. This is why the spectral isotropy, yielded by any of the H-type groups, is a very surprising phenomenon indeed. It puts a completely new light to the radiation isotropy evidencing the spatial-isotropy assumed in Friedmann's model. According to the above theorem, the radiation isotropy seems to be evidencing all the new relativistic models of elementary particle systems which are built up in this paper by nilpotent Lie groups and their solvable extensions. These models are far beyond those satisfying the spatial-isotropy assumption.

By these arguments, all the isospectrality examples established in [Sz1]-[Sz4] for a family $H_l^{(a,b)}$ defined by the same $a+b$ and l can be reestablished almost in the same way. Note that such a family is defined on the same (X, Z) -space and two members defined for (a, b) resp. (a', b') are not locally isometric, unless $(a, b) = (a', b')$ upto an order. The above intertwining operator proving the spectral isotropy appears now in the following modified form $\Omega_{Qpq\bullet} : \Xi_{Qpq\bullet} \rightarrow \Xi'_{Qpq\bullet}$, that is, it corresponds one-pole functions having the same pole but which are defined by the distinct complex structures J_{K_u} resp. J'_{K_u} to each other. The precise correspondence is then

$$(150) \quad \begin{aligned} \mathcal{H}\mathcal{F}_{Qpq\bullet}(\phi) &= \int_{\bullet} e^{i\langle Z, K \rangle} \phi(\mathbf{x}, K) \Pi_X^{(n)}(\Theta_Q^p \overline{\Theta}_Q^q)(X, K_u) dK_{\bullet} \rightarrow \\ &\rightarrow \mathcal{H}\mathcal{F}'_{Qpq}(\phi) = \int_{\bullet} e^{i\langle Z, K \rangle} \phi(\mathbf{x}, K) \Pi_X^{(n)}(\Theta_Q'^p \overline{\Theta}_Q'^q)(X, K_u) dK_{\bullet}, \end{aligned}$$

which, by the same argument used for proving spectral isotropy, intertwines both the Laplacian and the boundary conditions on any of the ball \times ball-resp. sphere \times ball-type domains.

In order to establish the complete isospectrality, pick the same system \mathbf{B} of independent vectors for both of these manifolds and, by implementing the obvious alterations in the previous formula, define $\Omega_{\mathbf{B}p_iq_i\bullet} : \Xi_{\mathbf{B}p_iq_i\bullet} \rightarrow \Xi'_{\mathbf{B}p_iq_i\bullet}$. This is a well defined operator between the complete L^2 function spaces which follows from the theorem asserting that the twisted Z-Fourier transforms are L^2 bijections mapping $\mathbf{P}\Phi_{\mathbf{B}}$ onto an everywhere dense subspace of the complete straightly defined space $\overline{\Phi}_{\mathbf{B}}$. It can be defined also by all those maps $\Omega_{Qpq\bullet}$ where pole Q is in the real span of the vector system \mathbf{B} . Thus, by the above argument, operators $\Omega_{\mathbf{B}p_iq_i\bullet}$ intertwine the complete L^2 function spaces along with the boundary conditions.

Interestingly enough, the isospectrality can be established also in a new way, by using only the intertwining operators $\omega_{Q\tilde{Q}pq\bullet}$. Indeed, suppose that the elements, $\{\nu_{p,q,i}\}$, of the spectrum appear on a total one-pole space Ξ_{Qpq} with multiplicity, say $m_{pq,i}$. By the one-pole intertwining operators this spectrum is uniquely determined and each eigenvalue regarding the whole L^2 -space must be listed on this list, furthermore, the multiplicity regarding the whole L^2 -space is the multiple of these one-pole multiplicities by the dimension of Ξ_{Bpq} . On the other hand, for $Q \in \mathbb{R}^{(l)a}$, the isospectrality obviously follows from $\Xi_{Qpq} = \Xi'_{Qpq}$, thus both the elements of the spectra and the regarding multiplicities must be the same on these two manifolds.

This proof clearly demonstrates that how can the spectrum “ignore” the isometries. It explains also the striking examples where one of the members of the isospectrality family is a homogeneous space while the others are locally inhomogeneous. It also demonstrates that spectral isotropy implies the isospectrality. Recall that this isospectrality is established above by an intertwining operator which most conspicuously exhibits the following so called C-symmetry principle of physics: “The laws remain the same if, in a system of particles, some of them are exchanged for their anti-particles.” This intertwining operator really operates such that some of the particles are exchanged for their anti-particles. Thus this proof demonstrates that spectral isotropy implies the C-symmetry. Moreover, the isospectrality is the manifestation of the physical C-symmetry. This is a physical verification of the isospectrality of the examples established in [Sz1]-[Sz4]. Actually these examples provide a rigorous mathematical proof for the C-symmetry which is not a theorem but one of the principles in physics. Let it be mentioned yet that the isospectrality proof provided in this paper is completely new, where the Hankel transform appears in the very first time in these investigations. (All the other proofs are established by different integral transformations.)

The isospectrality theorem naturally extends to the solvable extensions endowed with positive definite invariant metrics. Just functions $\phi(\mathbf{x}, K)$ should be exchanged for $\phi(\mathbf{x}, t, K)$ in the above formulas, that is, the intertwining is led back to the nilpotent group. It is important to keep in mind that the metrics are positive definite in the spectral investigations of the solvable extensions. The group of isometries acting on the sphere \times sphere-type manifolds of $SH_3^{(a,b)}$, where $ab \neq 0$, is $(\mathbf{Sp}(a) \times \mathbf{Sp}(b)) \cdot SO(3)$, while it is $\mathbf{Sp}(a+b) \cdot \mathbf{Sp}(1)$ on $SH_3^{a+b,0}$. By these formulas, the same statement proved for the nilpotent groups can be generalized to the solvable isospectrality family. These solvable examples provide also new striking examples of isospectral metrics where one of them is homogeneous while the other is locally inhomogeneous. By summing up, we have

Theorem 8.1. Operators Ω_{Qpq} and $\omega_{Q\bar{Q}pq}$ defined for combined spaces intertwine the Laplacians, moreover, they can be induced by point transformations of the form (K_Q, id_Z) resp. $(O_{Q\bar{Q}}, \text{id}_Z)$, where id_Z is the identity map on the Z -space and the first ones are orthogonal transformations on the X -space, transforming subspace S_Q , spanned by Q and all $J_{Z_u}(Q)$, to S'_Q resp. $S'_{\bar{Q}}$. This part of the map is uniquely determined, whereas, between the complement spaces it can be arbitrary orthogonal transformation.

By this induced map interpretation, both the Dirichlet and Z -Neumann conditions are also intertwined by these operators. This statement follows also from the fact that the very same operators are defined by corresponding the Hankel functions obtained by the Hankel decomposition of the above functions to each other. That is, these maps intertwine also the corresponding Hankel subspaces along with the exterior operator \mathbb{E} and the interior strong force operator \mathbf{S} .

So far the isospectrality is established for one-pole functions. For a global establishment consider a system \mathbf{B} of $k/2$ independent vectors described earlier on the X -space. Then the global operator $\Omega_{\mathbf{B}p_iq_i} : \mathcal{HF}_{\mathbf{B}p_iq_i}(\phi) \rightarrow \mathcal{HF}'_{\mathbf{B}p_iq_i}(\phi)$ can be defined also by Q -pole functions satisfying $Q \in \text{Span}_{\mathbb{R}}(\mathbf{B})$. This proves that the $\kappa_{\mathbf{B}p_iq_i}$ defines a global intertwining operator.

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